# Practical application of digital fractional-order controller to temperature control

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***Praktická aplikácia číslicového regulátora neceločíselného rádu na riadenie teploty***

*Príspevok sa zaoberá praktickým návrhom a aplikáciou číslicového regulátora neceločíselného rádu na riadenie teploty žiariča. V príspevku je ďalej uvedený matematický popis regulátorov a regulovaných sústav neceločíselného rádu. Pre použitie číslicových regulátorov neceločíselného rádu je uvedený príslušný algoritmus. Vlastnosti regulačného obvodu s regulátorom neceločíselného rádu sú porovnané s regulačným obvodom s klasickým regulátorom celočíselného rádu a dosiahnuté výsledky sú uvedené a diskutované v závere príspevku.*

***Kľúčové slová:*** *číslicový regulátor neceločíselného rádu, regulovaný systém neceločíselného rádu, analýza kvality riadenia*

## Introduction

The use of fractional calculus for modelling physical systems has been considered (Oldham et al., 1974; Torvik et al., 1984; Westerlund et al., 1994). We can find also works dealing with the application of this mathematical tool in control theory (Axtell et al., 1990; Dorčák, 1994; Outstaloup, 1995; Podlubny, 1994) but these works have only theoretical character. Works in which are analysed and described the real object and then designed and realised the fractional-order controller are still missing. Therefore we tried give example for practical application of the fractional-order controller on the real object and show obtained experimental results.

Motivation on using the digital fractional-order $PI^{\lambda}D^{\delta}$ controllers was that PID controllers belong to the dominating industrial controllers and therefore there is a continuous effort to improve their quality and robustness. The aim of this paper is also to show how to enhance quality of PID controllers. One of the possibilities to improve PID controllers is to use fractional-order $PI^{\lambda}D^{\delta}$ controllers with non-integer differentiation and integration parts.

In this paper we present a detailed analysis of the real object (heat solid) and a practical application of the fractional-order $PD^{\delta}$ controllers. We point out the non-adequate approximation of the non-integer systems by the integer-order models and differences in their closed loop behaviour.

## Fractional calculus

The fractional calculus is a generalisation of integration and derivation to non-integer order operator. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695. We use the generalisation of the differential and integral operators into one fundamental operator ${}_aD_t^{\alpha}$, where $a$ and $t$ are the limits related to the operation of fractional differentiation.

The two definitions used for fractional differentiation and integration are the Grünwald-Letnikov definition and the Riemann-Liouville definition. The Grünwald-Letnikov definition is given here (Oldham et al., 1974; Podlubny, 1999):

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \Delta_h^{\alpha} f(t), \tag{1} $$

where $[\ x\ ]$ means the integer part of $x$ and where $\Delta_h^{\alpha} f(t)$ is the generalised finite difference of order $\alpha$ with step $h$.

The Riemann-Liouville definition is given as (Oldham et al., 1974; Podlubny, 1999)

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$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2} $$

for ($n$ - 1 < $\alpha$ < $n$) and where $\Gamma(.)$ is the well known Euler's gamma function.

For a wide class of functions which appear in real physical and engineering applications, the Riemann-Liouville and the Grünwald-Letnikov definitions are equivalent (Podlubny, 1999).

The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of fractional derivative (2) has the form (Podlubny, 1999):

$$ L\left\{{}_0D_t^{\alpha} f(t)\right\} = p^{\alpha}F(p) - \left[{}_0D_t^{\alpha-1} f(t)\right]_{t=0}, \tag{3} $$

and for the integral:

$$ L\left\{{}_0D_t^{-\alpha} f(t)\right\} = p^{-\alpha}F(p). \tag{4} $$

For numerical calculation of fractional-order derivation we can use relation derived from the Grünwald-Letnikov definition (1). This relation has the following form (Dorčák, 1994; Podlubny, 1999):

$$ {}_{(t-L)}D_t^{\alpha} f(t) \approx h^{-\alpha} \sum_{j=0}^{N(t)} b_j \, f(t-jh), \tag{5} $$

where $L$ is the "memory length", $h$ is the step size of calculation and

$$ N(t) = \min\left\{\left[\frac{t}{h}\right], \left[\frac{L}{h}\right]\right\}, \tag{6} $$

where $[x]$ means the integer part of $x$. The binomial coefficient $b_j$ we can calculate from the following formula:

$$ b_0 = 1, \quad b_j = \left(1 - \frac{1+\alpha}{j}\right) b_{j-1}. \tag{7} $$

We can also use for computing of the binomial coefficient $b_j$ the relation based on the fast Fourier transform method described in work (Podlubny, 1999).

Other discrete approximation of fractional integration was given by Lubich where as the generating function was used the power series expansion of $(1-z^{-1})^{\alpha}$ . If we use this power series expansion we can obtain the formula for fractional integral of order $\alpha$ (Lubich, 1986):

$$ {}_0D_t^{-\alpha} f(t) = h^{\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha}{k} f(t-kh). \qquad 8) $$

Another possibility for the approximation is the use of the trapezoidal rule (Gorenflo, 1997) and then the power series expansion of the generating function for obtaining the coefficients and the form of the approximation as

$$ \left(\omega(z^{-1})\right)^{\pm\alpha} = \left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm\alpha}, \tag{9} $$

where $z$ is the complex variable and $z^{-1}$ is the shifting operator. The important thing is to have a limited number of coefficients because of the limited allowable memory of the microprocessor system.

**Fractional-order controlled systems and fractional-order controllers**

Let is consider a single input - single output feed-back control system with an unit gain in the feed-back loop (Fig.1), where W is required value, E is control error, U is control value and Y is real value.

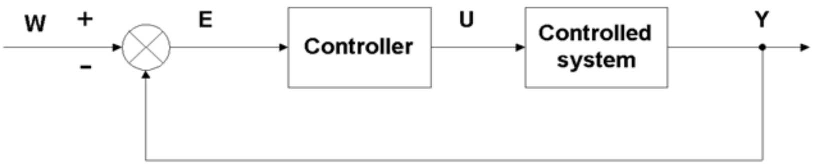


*Fig. 1 Feed - back control loop*

The fractional-order controlled system can be described by the fractional-order model with the differential equation of the form ( ${}_0D_t^{\mu} \equiv D_t^{\mu}$ ):

$$a_n\, D_t^{\alpha_n} y(t) + \ldots + a_1\, D_t^{\alpha_1} y(t) + a_0\, D_t^{\alpha_0} y(t) = b_m\, D_t^{\beta_m} u(t) + \ldots + b_1\, D_t^{\beta_1} u(t) + b_0\, D_t^{\beta_0} u(t), \qquad (10)$$

or by using Laplace transform methods (3) and on the equation (10), we have continuous transfer function of the form:

$$G_s(p) = \frac{b_m p^{\beta_m} + \ldots + b_1 p^{\beta_1} + b_0 p^{\beta_0}}{a_n p^{\alpha_n} + \ldots + a_1 p^{\alpha_1} + a_0 p^{\alpha_0}}. \qquad (11)$$

By doing the expression (11) and the discrete approximation of fractional integro-differential operator we can obtain a general expression for the discrete transfer function of the fractional system in form:

$$G_s(z) = \frac{b_m\left(\omega(z^{-1})\right)^{\beta_m} + \ldots + b_1\left(\omega(z^{-1})\right)^{\beta_1} + b_0\left(\omega(z^{-1})\right)^{\beta_0}}{a_n\left(\omega(z^{-1})\right)^{\alpha_n} + \ldots + a_1\left(\omega(z^{-1})\right)^{\alpha_1} + a_0\left(\omega(z^{-1})\right)^{\alpha_0}}, \qquad (12)$$

where $\left(\omega(z^{-1})\right)$ denote the discrete equivalent of the Laplace operator $p$ and where $\alpha_k$, $\beta_k$, ($k$ = 0, 1, 2, …, $n$) are generally real numbers, $\beta_n > \ldots > \beta_1 > \beta_0 \geq 0$, $\alpha_n > \ldots > \alpha_1 > \alpha_0 \geq 0$ and $a_k$, $b_k$, ($k$ = 0, 1, 2, …, $n$) are arbitrary constants.

The fractional-order controller $PI^{\lambda}D^{\delta}$ can be described by the fractional-order differential equation as (Podlubny, 1994)

$$u(t) = Ke(t) + T_i\, D_t^{-\lambda} e(t) + T_d\, D_t^{\delta} e(t), \qquad (13)$$

or by continuous transfer function of the form:

$$G_r(p) = K + T_i\, p^{-\lambda} + T_d\, p^{\delta}. \qquad (14)$$

The discrete approximation of the fractional-order controller can be expressed as

$$G_r(z) = K + T_i\left(\omega(z^{-1})\right)^{-\lambda} + T_d\left(\omega(z^{-1})\right)^{\delta}, \qquad (15)$$

where $\lambda$ is an integral order, $\delta$ is a derivation order, $K$ is a proportional constant, $T_i$ is an integration constant and $T_d$ is a derivation constant.

The fractional-order control system has a non-limited memory being the integer-order systems particular cases of this general case in which the memory is limited. It is obvious that only in the case of integer-order we could realise exactly a transfer function by using conventional lumped elements or procedures (finite order difference equations in the case of discrete realisation). For the practical realisation of the control algorithm we will use the equation (13) and the expression (5).

### Description and properties of controlled object

As a controlled object was chosen the laboratory object consist of the heat solid and the pyrometer. This object was connected to personal computer by programmable laboratory card PLC812 (Advantech Co.). The block diagram of connection is shown on Fig.2.

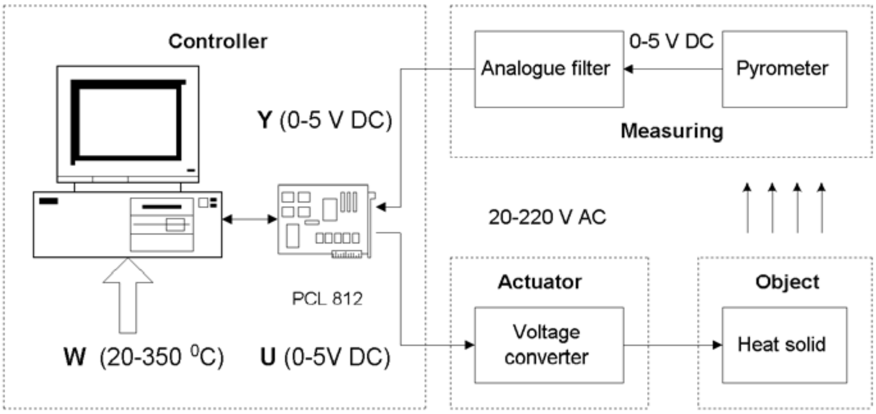


*Fig. 2 Block diagram of control*

The mathematical model of controlled object was obtained by the identification method given in (Dorčák et al., 1996). This method is based on the measured transient response of the unit step and to evaluate the quadratics criteria deviation between the measured values and the model values. The transient function of the controlled object (heat solid) obtained by experiment is shown in Fig.3.

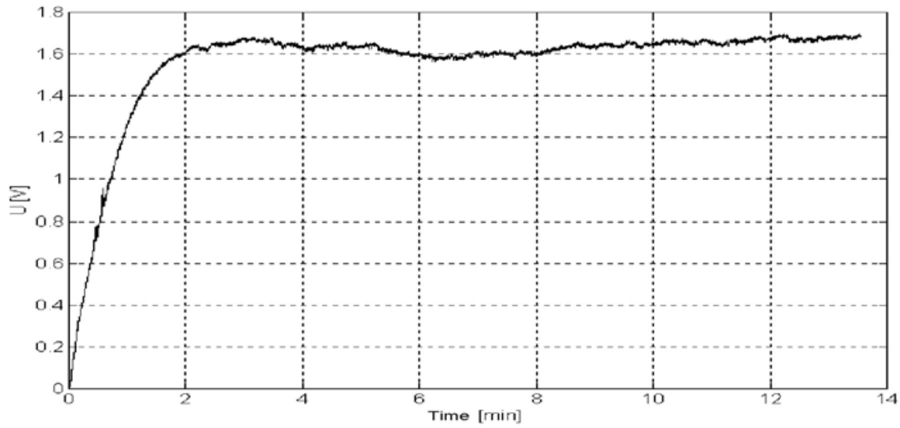


*Fig. 3 Transient unit response of the controlled object*

For the mathematical description of the controlled object was chosen two-term differential equation of the fractional-order. This mathematical description can be approximated by differential equation of first order.

The mathematical model of the controlled object described by two-tem differential equation has the form:

$$a_1 \, D_t^{\alpha} y(t) + a_0 y(t) = u(t). \tag{16}$$

For the differential equation (16) we can write the continuos transfer function as

$$G_s(p) = \frac{1}{a_1 p^{\alpha} + a_0}. \tag{17}$$

The coefficients of the transfer function and differential equation (16) was obtained by identification and have values:

$$a_1 = 39.69, \quad a_0 = 0.598, \quad \alpha = 1.26. \tag{18}$$

This real object with parameters (18) was approximated by the least square method the first order differential equation

$$\tilde{a}_1 y'(t) + \tilde{a}_0 y(t) = u(t) \tag{19}$$

with following values of the coefficients:

$$\tilde{a}_1 = 20.14, \quad \tilde{a}_0 = 0.598. \tag{20}$$

These mathematical models we will be use for controller design and for comparison of the control quality.

## Description and properties of controller

For the control quality comparison were chosen and designed the integer-order PD controller with the continuos transfer function:

$$G_r(p) = K + T_d p \tag{21}$$

and the fractional-order PD$^\delta$ controller with the continuos transfer function:

$$G_r(p) = K + T_d p^\delta. \tag{22}$$

The controller synthesis was done according to the method described in (Petráš et al., 1997). The design method is based on the known parameters of controlled object and required measure of stability and measure of damping. In the our case we designed the controllers for stability measure $S_t$ = 2.0. The integer-order PD controller (21) was designed to the integer-order mathematical model with the coefficients and the controller parameters have values:

$$\tilde{K} = 64.47, \quad T_d = 12.39. \tag{23}$$

The parameters of the fractional-order PD$^\delta$ controller designed to the fractional-order model with the coefficients (18) have values:

$$K = \tilde{K}, \quad T_d = 48.99, \quad \delta = 0.5. \tag{24}$$

Both of these controllers are the particular cases of the fractional-order PI$^\lambda$D$^\delta$ controller. The PI$^\lambda$D$^\delta$ controller (14) is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

The general discrete transfer function of the fractional-order PD$^\delta$ controller can be viewed as the Z version of the Grünwald-Letnikov formula (1) with the necessary use of the short memory principle proposed in (Podlubny, 1999). With sample period $T$ and memory length $L$ the discrete transfer function of the fractional-order PD$^\delta$ controller can be expressed as:

$$G_r(z) = K + T_d \frac{(T)^{-\delta} \sum_{k=0}^{\frac{L}{T}} (-1)^k \binom{\delta}{k} z^{\frac{L}{T}-k}}{z^{\frac{L}{T}}}. \tag{25}$$

We have two possibilities for realising a controller. The first is based on the use a physical device and the second is software or digital realisation based on program that will run in a computer or in a microprocessor.

## Control algorithm for fractional - order controllers

The control algorithm was designed according to the control scheme in Fig.1. The position algorithm, in generally form for $PI^{\lambda}D^{\delta}$ controller, uses discrete time steps *k* and consists of the following steps:

1. Filtering of required value:

$$w^*(k) = w^*(k-1) + 0.5(w(k) - w^*(k-1)), \tag{26}$$

where $w(k)$ is required value.

2. Calculating the control error:

$$e(k) = w^*(k) - y(k), \tag{27}$$

for discrete time step ($k$ = 1,2,...), where $e(k)$ is regulation error and $y(k)$ is measured value.

3. Determination of control value:

$$u(k) = Ke(k) + \frac{T_i}{T^{-\lambda}}\sum_{j=v}^{k} q_j\, e(k-j) + \frac{T_d}{T^{\delta}}\sum_{j=v}^{k} d_j\, e(k-j), \tag{28}$$

for discrete time step ($k$ = 1, 2, …), where $T$ is the length of time step (sample period). The binomial coefficients $d_j$ and $q_j$ were calculated from the recurrent equation. The numerical algorithm requires to store the whole history ($v$ = 0). For improving their effectiveness we have used "short memory" principle, where $v = 0$ for $k < (L/T)$ or $v = k - (L/T)$ for $k > (L/T)$. Besides the "short memory" the control quality is influenced by time step $T$.

The sample period is very important parameter in the control algorithm. For good appropriation of the sample period we can use references in work (Petráš et al., 1999).

## Comparison of fractional and integer - order controllers

The comparison of the control quality was done by applying both controllers to the real object. For the control quality arbitration were chosen the following criteria: control surface - *Sr*; overshoot - *Pr*; settling time - *Tr*. We measured the values of these criteria for total time 10 [*min*] for unit change of required value from 0[V] to 1[V]. In Tab. 1 are results from regulation of the controlled object by the integer and the fractional-order controllers. Transfer relation (linear) between voltage U [V] and temperature t [$^0$C] can be expressed as

$$t = U(330/5) + 20. \tag{29}$$

The controller in our case was realised as a software programmed in the computer language Pascal 6.0, running on the personal computer with card PCL 812. The control algorithm was programmed in according to equations (26) – (28) and was modified for $PD^{\delta}$ controller $T_i = 0$ ($\lambda$ =0). The sample period was used $T$ = 1 [*s*] and the length of controller memory was $L$=100 samples.

On the Fig.4. is shown the comparison of the unit-step responses the integer-order PD controller applied on the fractional-order system (thin line) and the fractional-order $PD^{\delta}$ controller applied on the fractional-order system (thick line). The unit-step responses were obtained by the control algorithm described above with designed parameters for PD and $PD^{\delta}$ controllers.

*Tab. 1 The results from regulation for 10* [*min*]

| - | PD - (18) | $PD^{\delta}$ - (18) |
|---|---|---|
| *Sr* [-] | 2.4423 | 1.7469 |
| *Pr* [%] | 10.25 | 3.94 |
| *Tr* [*s*] | 3.08 | 1.01 |

By applying the integer-order controller to the original system we do not achieve the required quality of the regulation process as with the approximated integer-order system. This was confirmed via simulation in the time domain (Dorčák, 1994; Podlubny, 1994) and also by checking the stability measure and damping measure (Petráš et al., 1997).

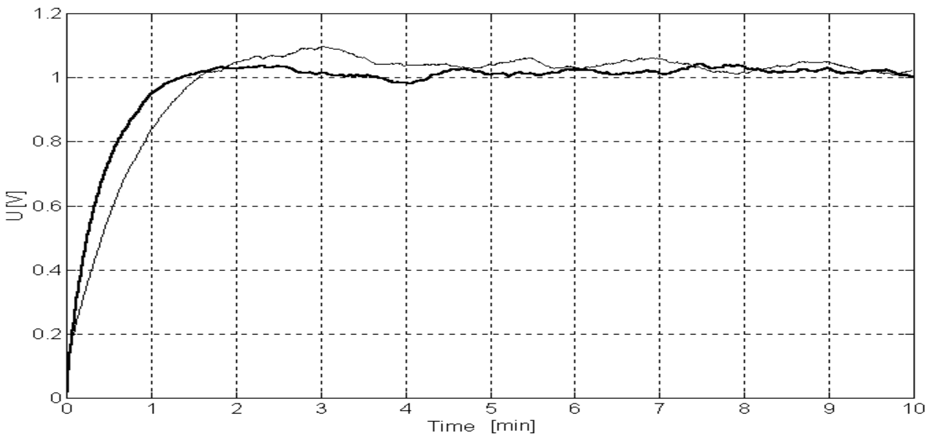


*Fig. 4 Transient unit response of the control loop*

## Conclusion

The dynamical properties of the closed loop with the fractional-order controlled system and the fractional-order controller are better than the dynamical properties of the closed loop with the integer-order controller. The system with the integer-order controller stabilises slower and has larger surplus oscillations. We can see that the use of the fractional-order controller leads to the improvement of the control of the real system.

For further work we plan realisation of an analogue controller by RC networks and operating amplifiers. For practical realisation of this controller we will use the results from works (Petráš et al., 1999; Westerlund et al., 1994). The analogue controller will be compared with the digital controller realised at this work.

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