Global theory of nonlinear systems-chaos, knots and stability

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Globálna teória nelineárnych systémov-Chaos, uzly a stabilita

In this paper we shall give a brief overview of nonlinear dynamical systems theory including the theory of chaos, knots, approximation theory and stability.

***Key words:** Chaos, knots, approximation theory, stability.*

Introduction

Ever since the fundamental ideas of Poincaré in the qualitative theory of dynamical systems, there has been intense interest into the global behaviour of such systems. Many important ideas and examples have grown out of the basic theory including chaos, knotted trajectories, fractal dimension and equivalence theory. Here we shall give an overview of some of the most important results which are crucial to the understanding of the behaviour of such dynamical systems, including chaos, knots and links, approximation techniques, and stability.

Chaos

Chaos in dynamical systems has been studied for some time now and although no universally accepted definition of chaos seems to exist, the best definition is in terms of systems with fractal Poincaré sections in their attractors, generating discrete symbolic systems with 'arbitrarily complex behaviour'. To see what this means, first consider the discrete dynamical system defined on a subset Δ of the closed unit square given by the product of two Cantor sets as in fig. 1.

The dynamics are defined by the two affine functions shown in the figure, which operate on the top and bottom thirds of the square. Since we have removed the middle thirds of the intervals in the horizontal and vertical directions, we can code the points of the 'Cantor square' by a bidirectional binary sequence

$$b = \ldots\ b_{-2}b_{-1} \cdot b_0 b_1 b_2 \ldots$$

(for details, see Wiggins, 1988). The space of all such sequences Σ can be made into a metric space by the distance

$$d(b,b') = \sum_{i=-\infty}^{\infty} \frac{\delta_i}{2^{|i|}} \quad \text{, where } \delta_i = \begin{cases} 0 & \text{if } b_i = b'_i \\ 1 & \text{if } b_i \neq b'_i \end{cases}.$$

On this metric space we can define a 'symbolic dynamical system' by the shift map:

$$\sigma(\ldots\ b_{-2}b_{-1} \cdot b_0 b_1 b_2 \ldots) = \ldots\ b_{-2}b_{-1}b_0 \cdot b_1 b_2 \ldots\ .$$

It can be shown that f and σ are conjugate dynamical systems and so we can study f by analysing σ. The complexity of σ can be seen by noting that it has

(i) a countable infinity of periodic orbits of arbitrary period
(ii) an uncountable infinity of nonperiodic orbits
(iii) a dense orbit.

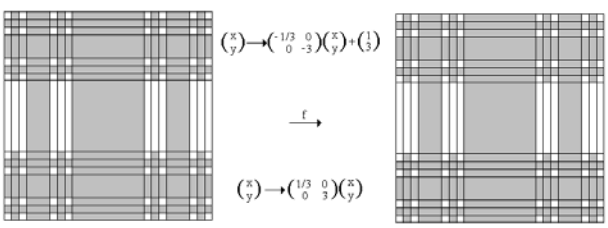

Fig.1. The Smale Horseshoe.

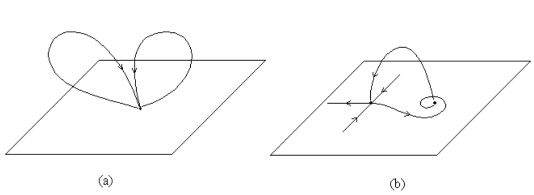

Fig.2. (a) System with two homoclinic orbits (b) A heteroclinic cycle.

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A chaotic dynamical system can be defined as one which contains a Poincaré section on which the dynamics is conjugate to a (sub)system of type (σ,Σ). A typical situation in three-dimensions is when a system contains a homoclinic orbit or a heteroclinic cycle as shown in fig. 2 (see Shil'nikov, 1970).

Knots and Links in Dynamical Systems

Given a nonlinear system with periodic orbits we are interested in when they are knotted and linked together. First, the formal definitions.

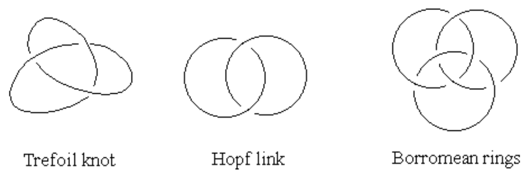

Fig.3. Simple Knots and Links.

Definition A **knot** is an embedding of S^1 (the 1-dimensional sphere, i.e. a circle) into S^3 (the 3-sphere). A **link** is an embedding of a disjoint union of circles. Simple examples of knots and links are shown in fig. 3.

These two-dimensional diagrams are called **regular knot projections**. An important question is : when are two knots or links equivalent (i.e. the same)?

Definition Two knots K and K' are (ambient) **isotopic** if there exists a continuous homotopy $h_t : S^3 \to S^3$ of homeomorphisms such that $h_0 =$ id and $h_1 \circ K = K'$.

A very difficult question is – when are two knots isotopic? Reidmeister (Reidemeister, 1932) showed that (apart from an ambient isotopy preserving the crossing points in regular graphs such as those in fig. 3), two regular knot projections correspond to isotopic links if and only if they are related by a finite sequence of the following three Reidemeister moves (shown in fig. 6).

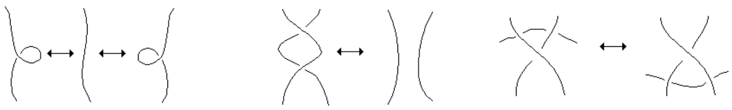

Fig.4. The Reidemeister Moves.

Consider now systems defined

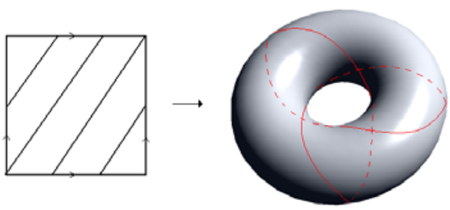

Fig.5. A Trefoil Knot on the T orus.

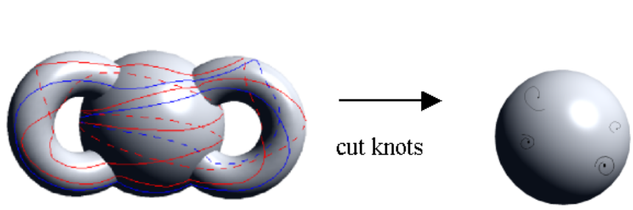

Fig.6. Surface of genus 2 with Two Distinct Knots and the Corresponding Spherical System.

on two-dimensional compact surfaces. If S is a surface of genus p then it can carry a dynamical system with many nontrivial knots – e.g. a torus can contain any torus knot defined by opening out the torus as in fig. 5.

However we can ask – how many topologically distinct (non-homotopically trivial) knots can a surface of genus p carry and what does the rest of the dynamics look like? This is answered in Banks (Banks, 2002) and the answer is simple – a surface of genus p can carry only p distinct knots. By cutting the surface along the knots and shrinking them to zero it can also be shown that the remaining dynamics is equivalent to any spherical dynamics containing at least $2p$ equilibria of index +1. (See fig. 6.)

Approximation Theory

In this section we shall introduce a sequence of linear, time-varying approximations for solving the nonlinear problem

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t),\ x(0) = x_0 \in \mathbf{R}^n \tag{4.1}$$

(If $B \equiv 0$, we have a standard continuous-time dynamical system, while if $B \neq 0$ then we have a nonlinear control system. We can consider partial differential equations and delay systems in the same way.) The approximations are given by

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t),\ x^{[i]}(0) = x_0\ (i \geq 1) \tag{4.2}$$

and

$$\dot{x}^{[0]}(t) = A(x_0)x^{[0]}(t) + B(x_0)u^{[0]}(t),\ x^{[0]}(0) = x_0 .$$

It can be shown ([4]) that the solutions of the sequence of unforced systems

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t),\ x^{[i]}(0) = x_0 \tag{4.3}$$

converges uniformly on compact time intervals to the solution of the nonlinear system

$$\dot{x}(t) = A(x(t))x(t),\ x(0) = x_0 \quad . \tag{8.4}$$

For example, a selection of the first 14 iterates of the Van der Pol equation

$$\dot{x}_1 = x_1 - x_1^3 + x_2$$

$$\dot{x}_2 = -x_1$$

given by the system

$$\begin{pmatrix} \dot{x}_1^{[i]}(t) \\ \dot{x}_2^{[i]}(t) \end{pmatrix} = \begin{pmatrix} -(x_1^{[i-1]}(t))^2 + 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1^{[i]}(t) \\ x_2^{[i]}(t) \end{pmatrix}$$

are shown in fig. 7. For stability theory see (Tomas-Rodriguez and Banks, 2003).

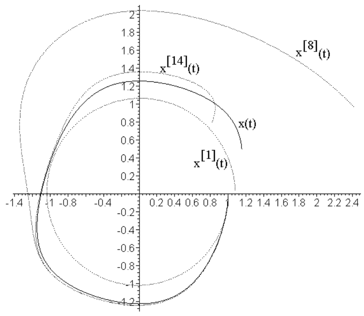

Fig.7. Iterates of the Van der Pol Oscillator.

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