Second-order sliding mode control of time delay systems

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Riadenie systémov s časovým oneskorením druhého rádu v kĺzavom móde

The contribution of the paper is a proposal of the second-order sliding mode control of time delay systems developed analogously to the first-order version. Unlike the control variable itself in the first-order version the first derivative (speed) of this variable is assessed in the second-order scheme of the SMC (Sliding Mode Control). In accordance with the first order SMC scheme a parallel model of the control system is applied prescribing its desired behaviour, to be followed by the controlled process. The model of the feedback system consists of both the process and the feedback controller models adjusting the control action amplitude. The process model is supposed to be identified as time delay system described by an anisochronic state model, that brings a low order form advantageous for the control synthesis. This synthesis is based on a dominant pole assignment performed by the help of the Ackermann formula extension. The mismatch between the process and its model, known as perturbation, is rejected by sliding mode technique. The second-order sliding mode is proposed to suppress the usually encountered "chattering" of the first-order version of the sliding mode control.

Key words: *second-order sliding mode, time delay system, anisochronic model, Ackermann formula, perturbation, chattering, SMC scheme.*

Introduction

The idea of a higher-order instead of the first-order sliding mode control was originally proposed by (Levant, 1993), and later worked out by (Fridman et al., 1996; Levant, 2000). The second-order sliding mode control design for controlling time delay systems is the main aim of this paper. The first-order sliding mode control of time delay systems is designed for time delay system description with delays of distributed type in (Drakunov et al., 1992; Zheng et al., 1995). The application to sliding mode control of time delay systems identified as anisochronic systems is presented in (Zítek et al., 2001). Anisochronic state model used for the second-order sliding mode control design also includes distributed delays, in general, and just anisochronic state feedback combined with the sliding mode principle is the main contribution of (Zítek et al., 2001). The benefit of this combination consists in an insensitivity to controlled system perturbations and in achieving the desired feedback system dynamics (Zítek et al., 2001). A reduction of the sensitivity to system perturbation by the sliding mode control is largely discussed in (Jalili et al., 1998; Moura et al., 1997) where the perturbation is considered as parameter and structure uncertainties and external disturbances. The drawback known as "chattering" is significantly reduced just by the second-order SMC scheme, due to integrator filtering (Bartolini et al., 1998). The second-order sliding mode control algorithm is defined and then it is connected with an anisochronic state feedback.

Second-order sliding mode control

The second-order sliding mode control algorithm is defined by a discontinuous function based on the following switching function (Levant 2000)

$$\frac{du(t)}{dt} = \left\langle \begin{array}{ll} \left[\dfrac{du(\boldsymbol{x})}{dt}\right]^{+}, & \dfrac{dm(\boldsymbol{x})}{dt} m(\boldsymbol{x}) \geq 0 \\ \left[\dfrac{du(\boldsymbol{x})}{dt}\right]^{-}, & \dfrac{dm(\boldsymbol{x})}{dt} m(\boldsymbol{x}) < 0 \end{array} \right. , \quad \forall\, t \tag{1}$$

where: u is a control variable and $S = \left\{ \boldsymbol{x} \in \mathbf{R}^{n} : m(\boldsymbol{x}) = 0 \right\}$ is a given surface of sliding motion or in other words desired surface of discontinuity. The variable $u(t)$ is the control variable with a piece-wise continuous derivative given by (1). If the sliding surface $m(\boldsymbol{x})$ satisfies the condition (Utkin, 1992)

$$\frac{dm(t)}{dt} m(t) < 0, \quad \forall t \tag{2}$$

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where: $m(t)$ results from $m(\boldsymbol{x})$ for a certain $\boldsymbol{x} = \boldsymbol{x}(t)$, the sliding motion is stable. The first derivative of u is the unknown variable, that is assessed by a simultaneous application of the following conditions (Levant, 2000)

$$\frac{d^2 m(t)}{dt^2} = 0 \tag{3}$$

$$\frac{dm(t)}{dt} = 0 \tag{4}$$

Control variable u is limited by the interval

$$u^- < u(u_a) < u^+ \tag{5}$$

where: u_a is an auxiliary variable adjusted by a model introduced below (Zítek et al., 2001).

Thus the derivative (1) is dependent on the time derivative of $u_a(t)$

$$\frac{du}{dt} = f\left(\frac{du_a}{dt}\right) \tag{6}$$

and this derivative is bounded by the values

$$\left[\frac{du}{dt}\right]^- \equiv -U, \qquad \left[\frac{du}{dt}\right]^+ \equiv U \tag{7}$$

Both the magnitude U and the surface $m(\boldsymbol{x})$ are specified in chapter 3.

Second-order sliding mode control in anisochronic state space

The identified process model in the functional state space formulation (Zítek et al., 2001)

$$\frac{d\boldsymbol{x}(t)}{dt} = \int_0^T d\mathbf{A}(\tau)\boldsymbol{x}(t-\tau) + \int_0^T d\,\mathbf{b}(\tau)\,u(t-\tau) + \mathbf{f}(\boldsymbol{x},t) \tag{8}$$

with the output equation

$$y(t) = \mathbf{c}\boldsymbol{x}(t) \tag{9}$$

and with the vector of the perturbations $\mathbf{f}$ of dimension $n \times 1$ is applied to generate the desired state trajectory $\boldsymbol{x}^w$ by a parallel model

$$\frac{d\boldsymbol{x}^w(t)}{dt} = \int_0^T d\mathbf{A}(\tau)\boldsymbol{x}(t-\tau) + \int_0^T d\,\mathbf{b}(\tau)\,u_a(t-\tau) \tag{10}$$

where: the vectors $\boldsymbol{x}$ and $\boldsymbol{x}^w$ are of the same dimension $n \times 1$ and all of them are supposed to be measurable. With respect to functional nature of (8) and (10) it is advantageous to perform *L*-transform of (8) and (10) under zero initial conditions as follows

$$s\boldsymbol{x}(s) = \mathbf{A}(s)\boldsymbol{x}(s) + \mathbf{b}(s)u(s) + \mathbf{f}(s) \tag{11}$$

and

$$s\boldsymbol{x}^w(s) = \mathbf{A}(s)\boldsymbol{x}(s) + \mathbf{b}(s)u_a(s) \tag{12}$$

The matrices $\mathbf{A}(s), \mathbf{b}(s)$ are of the functional form

$$\mathbf{A}(s) = \int_0^T \exp(-s\tau)d\mathbf{A}(\tau), \quad \mathbf{b}(s) = \int_0^T \exp(-s\tau)d\mathbf{b}(\tau) \tag{13}$$

with dimensions (n,n) and $(n,1)$ respectively. The matrices (13) are s-multiples of ordinary $\mathbf{A}(\tau), \mathbf{b}(\tau)$ transforms. A reference control variable u_a is generated in this model assessed as an anisochronic state feedback (Zítek et al., 1999) as follows

$$u_a(t) = \int_0^T d\mathbf{k}(\tau)\left[\boldsymbol{x}_w(t-\tau) - \boldsymbol{x}(t-\tau)\right] \tag{14}$$

or in the *L*-transform

$$u_a(s) = \mathbf{k}(s)\left[\boldsymbol{x}_w(s) - \boldsymbol{x}(s)\right] \tag{15}$$

where: $\boldsymbol{x}_w$ is a reference state vector. With respect to relation (1) the second-order sliding mode control needs to specify the first derivative of the variable u_a, i.e. $\frac{du_a}{dt}$, and thus equation (15) is modified to the form

$$su_a(s) = \mathbf{k}(s)\left[s\boldsymbol{x}_w(s) - s\boldsymbol{x}(s)\right]. \tag{16}$$

The matrix $\mathbf{k}(s)$ can be obtained by the help of Ackermann formula extension, which performs a dominant pole assignment (Zítek et al., 1999). After substituting *L*-transform (11) of the process model (8) into (16) one can rewrite (16) as follows

$$su_a(s) = \left[\mathbf{b}(s)\mathbf{k}(s) - \mathbf{k}(s)\mathbf{A}(s)\right]\boldsymbol{x}(s) - \mathbf{k}(s)\mathbf{b}(s)\mathbf{k}(s)\boldsymbol{x}_w(s) \tag{17}$$

when assuming the first derivative of the reference state vector as $s\boldsymbol{x}_w(s) = 0$. The first derivative of u_a at time t is given by inverse *L*-transform of (17) as follows

$$\frac{du_a(t)}{dt} = \int_0^T d\left[\mathbf{b}(\tau)\mathbf{k}(\tau) - \mathbf{k}(\tau)\mathbf{A}(\tau)\right]\boldsymbol{x}(t-\tau) - \int_0^T d\left[\mathbf{k}(s)\mathbf{b}(s)\mathbf{k}(s)\right]\boldsymbol{x}_w(t-\tau). \tag{18}$$

Since the models (8) and (10) are introduced, the surface of the sliding motion derived in (Zítek et al., 2001) as follows

$$m(t) = \mathbf{c}\left[\boldsymbol{x}^w(t) - \boldsymbol{x}(t)\right] \tag{19}$$

can be applied to the condition (3). Therefore the second derivative of the state vector $\boldsymbol{x}$ as well as state trajectory $\boldsymbol{x}^w$ are found out using models (8) and (10) as follows

$$\frac{d^2\boldsymbol{x}(t)}{dt^2} = \int_0^T d\mathbf{A}^2(\tau)\boldsymbol{x}(t-\tau) + \int_0^T d\left[\mathbf{A}(\tau)\mathbf{b}(\tau)\right]u(t-\tau) + \int_0^T d\mathbf{b}(\tau)\frac{du_a(t-\tau)}{dt} + \int_0^T d\mathbf{A}(\tau)\,\mathbf{f}(t-\tau) + \mathbf{f}(t) \tag{20}$$

and

$$\frac{d^2\boldsymbol{x}^w(t)}{dt^2} = \int_0^T d\mathbf{A}^2(\tau)\boldsymbol{x}(t-\tau) + \int_0^T d\left[\mathbf{A}(\tau)\mathbf{b}(\tau)\right]u_a(t-\tau) + \int_0^T d\mathbf{b}(\tau)\frac{du_a(t-\tau)}{dt} \tag{21}$$

Instead of these functional equations their *L*-transforms are applied in the form

$$s^2\boldsymbol{x}(s) = \mathbf{A}^2(s)\boldsymbol{x}(s) + \mathbf{A}(s)\mathbf{b}(s)u(s) + \mathbf{b}(s)su(s) + \mathbf{A}(s)\mathbf{f}(s) + \mathbf{f}(s) \tag{22}$$

and

$$s^2\boldsymbol{x}^w(s) = \mathbf{A}^2(s)\boldsymbol{x}(s) + \mathbf{A}(s)\mathbf{b}(s)u_a(s) + \mathbf{b}(s)su_a(s) \tag{23}$$

After substituting the second derivatives (22) and (23) into *L*-transform of the condition (3) applied to (19) one can write

$$\mathbf{c}\left[\mathbf{A}(s)\mathbf{b}(s)\left[u_a(s) - u(s)\right] + \mathbf{b}(s)\left[su_a(s) - su(s)\right] - \mathbf{A}(s)\mathbf{f}(s) - \mathbf{f}(s)\right] = 0 \tag{24}$$

Then, on assumption that $\mathbf{c} \neq \mathbf{0}_{1\times n}$, it results from the condition (4)

$$\mathbf{b}(s)\left[u_a(s) - u(s)\right] - \mathbf{f}(s) = 0 \tag{25}$$

Consequently the relation (24) can be rewritten in the form

$$\mathbf{b}(s)\left[su_a(s) - su(s)\right] - \mathbf{f}(s) = 0 \tag{26}$$

The first derivative of the control variable is extracted from (26) as follows

$$su(s) = su_a(s) - \mathbf{b}_L^{-1}(s)\mathbf{f}(s) \tag{27}$$

where: $\mathbf{b}_L^{-1}(s) = \left[\mathbf{b}^T(s)\mathbf{b}(s)\right]^{-1}\mathbf{b}^T(s)$ is the left inverse of the matrix $\mathbf{b}(s)$. The inverse *L*-transform of (27) is as follows

$$\frac{du(t)}{dt} = \frac{du_a(t)}{dt} - \int_0^T d\left[\mathbf{b}_L^{-1}(\tau)\right]\mathbf{f}(t-\tau) \tag{28}$$

where: the first derivative of u_a is computed by (18). So the magnitude (7) is specified due to (6) and (28) as follows

$$U(t) > \left|\frac{du_a(t)}{dt}\right| + \gamma \tag{29}$$

where: the parameter γ is the so-called perturbation parameter (Fišer et al., 2002). This parameter is adjusted in the manner

$$\gamma > \left|\int_0^T d\left[\mathbf{b}_L^{-1}(\tau)\right]\mathbf{f}(t-\tau)\right| \tag{30}$$

as it is selected from (28) and (29). The inequality (30) is only accepted in the interval $\gamma \in \left(0, \Psi\right)$ (Fišer et al., 2002) where

$$\Psi = \sup(\| \mathbf{f} \|) \tag{31}$$

The perturbation $\mathbf{f}$ of the process model (8) is not considered in the first derivative (18) of the variable u_a, as the vector of perturbations $\mathbf{f}$ is explicitly expressed in the first derivative (28) of the control variable u. If all the state variables in the vector $\boldsymbol{x}$ are measurable they can be compared with the generated state vector trajectory $\boldsymbol{x}^w$. Due to the integration in the sliding mode feedback a favourable filtering effect is obtained in the control action. In addition the known advantage of the control algorithm it is easy way to remove a wind-up effect but also to reduce the chattering. by limited integrator.

Conclusions

The second-order sliding mode control of time delay systems based on anisochronic model has been developed as a modification of the usual scheme of SMC. In addition, some of the delay effects are compensated by the parallel model designed as anisochronic one. Unfortunately, the synthesis of model state feedback deriving the reference state trajectory may sometimes result in unfeasible feedback operator $\mathbf{k}(s)$. Physical feasibility can then be achieved by using Smith based control scheme. In the following research the features of the second-order sliding mode control, i.e. the chattering reduction and the insensitivity to perturbations, will be tested on a laboratory heat transfer plant with significant hereditary properties.

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