# Contribution to stability analysis of nonlinear control systems

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**_Príspevok ku analýze stability nelineárnych riadiacich systémov_**

*The Popov criterion for the stability of nonlinear control systems is considered. The Popov criterion gives sufficient conditions for stability of nonlinear systems in the frequency domain. It has a direct graphical interpretation and is convenient for both design and analysis. In the article presented, a table of transfer functions of linear parts of nonlinear systems is constructed. The table includes frequency response functions and offers solutions to the stability of the given systems. The table makes a direct stability analysis of selected nonlinear systems possible. The stability analysis is solved analytically and graphically.*

*Then it is easy to find out if the nonlinear system is or is not stable; the task that usually ranks among the difficult task in engineering practice.*

**Key words:** *Nonlinear control system; Popov criterion; modified frequency response; global asymptotic stability (GAS).*

The Popov criterion is considered as one of the most appropriate criteria for nonlinear systems and it can be compared with the Nyquist criterion for linear systems. However there are reservations that relate to the very essence, correctness and reliability of the criterion. It is necessary to emphasise that this criterion is reliable, but the conditions of its application should be clearly specified in advance.

Though, in this contribution, practical applications of the criterion are mostly dealt with. For all that I indicate the basic relations and conditions of its applications.

The derivation of the criterion can be found e.g. in (Kotek – Razim, 1988). Once again it is necessary to emphasise that the most general form of this criterion is not presented in this article (e.g. for two nonlinearities or nonlinearities with hysteresis). The form given is a common standard used in practice.

Figure 1 shows the system configuration with one nonlinear element and a linear part with the transfer function *G(s)* that can include all linear elements.

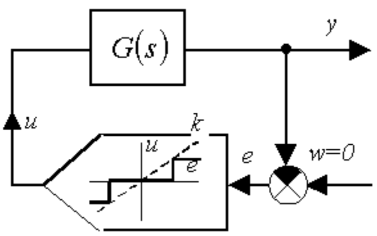


*Fig.1. Nonlinear control circuit.*

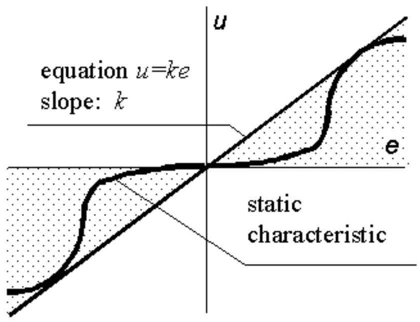


*Fig.2. Non-linear characteristic of nonlinear element.*

The nonlinearity is illustrated in figure 2. It is single-valued, time – invariant and constrained to a hatch sector bounded by slopes *k* that is assumed to satisfy

$$0 \leq \frac{f(e)}{e} \leq k < \infty \qquad (1)$$

for the case when all poles of *G(s)* are inside the left-half plane or

$$0 < \frac{f(e)}{e} \leq k < \infty \qquad (2)$$

for the case when *G(s)* has poles on the imaginary axis (the so-called critical case).

The nonlinearities considered are the following: for $e = 0$ is

$$f(e) = 0 \qquad (3)$$

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For the case that $G(s)$ has poles only inside the left-half $s$-plane, the static characteristic can be zero both in the beginning and out of the beginning . For the case when $G(s)$ has poles on the imaginary axis, it must not be zero out of the beginning.

The additional assumption is the limit stability of the nonlinear system. The nonlinear system is considered as limit stable on the condition that it is stable for the case when the nonlinear element is substituted by the linear element with an arbitrary small gain $\sigma > 0$.

A nonlinear circuit with a transfer function of the linear part $G(s)$ and with the nonlinear element (with the above described nonlinearity) is globally asymptotically stable when an arbitrary real number $q$ ($> 0$ or $= 0$ or $< 0$) exists where for every $\omega \geq 0$ the following inequality is completed

$$Re\left[(1 + j\omega q)G(j\omega)\right] + \frac{1}{k} > 0 \tag{4}$$

The Popov criterion can be – for more convenience – applied graphically in the $G(j\omega)$-plane. Let us apply a modified frequency response function $G^*(j\omega)$, defined by

$$Re\, G^*(j\omega) = Re\, G(j\omega)\ ; \qquad Im\, G^*(j\omega) = \omega . Im\, G(j\omega)\ ; \qquad \omega \geq 0 \tag{5}$$

and we obtain the graphical interpretation of the Popov criterion for global asymptotic stability (GAS):

The sufficient condition for GAS of nonlinear circuit is that the plot of $G^*(j\omega)$ should lie entirely to the right of the Popov line which crosses the real axis at $-1/k$ at a slope $1/q$ ($q$ is an arbitrary real number) – figure 3.

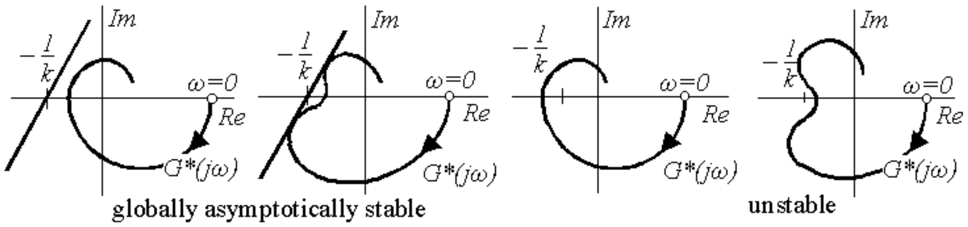


*Fig.3. Graphical interpretation of the Popov criterion.*

In this contribution, table 1 shows the commonly used nonlinear circuits (with stability being solved). The table has been constructed for the circuits with different transfer function $G(s)$.There is an algebraic solution to the inequality (4) on condition that:

$0 \leq k < \infty$ (only for poles $G(s)$ inside the left-half of $s$-plane); $0 < k < \infty$ (also for poles $G(s)$ on the imaginary axis); $a, b > 0$; $\omega$... for every value from $0$ to $\infty$ ; $q$...arbitrary.

In this table, the frequency response function of the linear part of the circuit is presented in the form inserted to the inequality (4), and the resultant relation is presented after modifications. If the inequality is satisfied for $k, q, a, b, \omega$ , the circuit in question is globally asymptotically stable (GAS) for every value $k$. If it is not the case, then it is possible to determine for which $k$ the circuit is GAS or that it is not GAS for any $k$ (this calculation is illustrated in the table). Therefore the table will allow us to directly determine the stability of the nonlinear circuit with the transfer function $G(s)$ and the nonlinearity that satisfies the slope $k$. The table can be further extended (additional $G(s)$) because this contribution is limited by space.

Figure 4 illustrates the modified frequency response plots enabling us graphic solutions to stability.

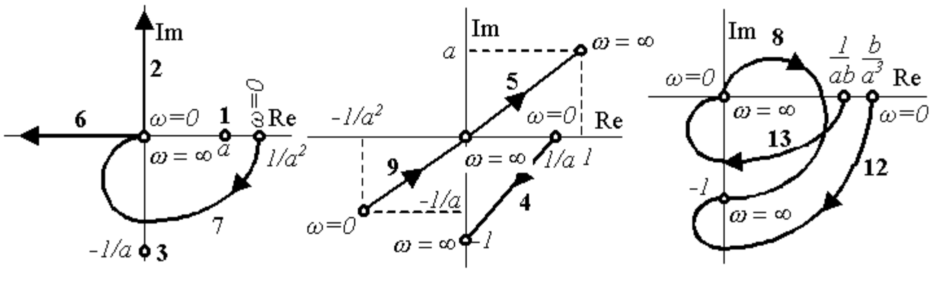


*Fig.4. Modified frequency response plots of nonlinear circuits.*

*Tab.1. Solutions to stability selected nonlinear circuits.*

| | *G(s)* | *G(jω)* in inequality (4) $Re[(1+j\omega q)G(j\omega)]+\frac{1}{k}>0$ | solution of (4) | conditions of stability |
|---|---|---|---|---|
| 1 | $a$ | $a$ | $ka+1>0$ | GAS for every $k$ |
| 2 | $as$ | $aj\omega$ | $1 - k\omega^3 q > 0$ | GAS for every $k$ |
| 3 | $\frac{1}{as}$ | $-j\frac{1}{a\omega}$ | $qk + a > 0$ | GAS for every $k$ |
| 4 | $\frac{1}{s+a}$ | $\frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$ | $k(a+\omega^2 q)+a^2+$ $+\omega^2>0$ | GAS for every $k$ |
| 5 | $\frac{s}{s+a}$ | $\frac{\omega^2}{a^2+\omega^2} + j\frac{a\omega}{a^2+\omega^2}$ | $k\omega^2(1-aq)+a^2+$ $+\omega^2>0$ | GAS for every $k$ |
| 6 | $\frac{1}{as^2}$ | $-\frac{1}{a\omega^2}$ | $k < a\omega^2$ | not stable for arbitrary $k$ |
| 7 | $\frac{1}{(s+a)^2}$ | $\frac{a^2-\omega^2}{(a^2+\omega^2)^2} - j\frac{2a\omega}{(a^2+\omega^2)^2}$ | $k\omega^2(2aq-1)+ka^2+$ $+(a^2+\omega^2)^2>0$ | GAS for every $k$ |
| 8 | $\frac{s}{(s+a)^2}$ | $\frac{2a\omega^2}{(a^2+\omega^2)^2} + j\frac{a^2\omega-\omega^3}{(a^2+\omega^2)^2}$ | $k\omega^2[2a+q(\omega^2-a^2)]+$ $+(a^2+\omega^2)^2>0$ | GAS for every $k$ |
| 9 | $\frac{1}{s(s+a)^2}$ | $\frac{-2a}{(a^2+\omega^2)^2} + j\frac{\omega^2-a^2}{\omega(a^2+\omega^2)^2}$ | … | GAS for $k < 2a^3$ |
| 10 | $\frac{1}{(s+a)^3}$ | $\frac{1}{(j\omega+a)^3} = \ldots\ldots$ | … | GAS for $k < 8a^3$ |
| 11 | $\frac{s+b}{s(s+a)^2}$ | $\frac{j\omega+b}{j\omega(j\omega+a)^2} = \ldots$ | GAS for $k > -\frac{4a^6-12a^5b+6a^4b^2}{4a^4-10a^3b+8a^2b^2-2ab^3}$ | |
| 12 | $\frac{1}{(s+b)(s+a)^2}$ | $\frac{1}{(j\omega+b)(j\omega+a)^2} = \ldots$ | GAS for $k < \frac{2a^5+8a^4b+12a^3b^2+8a^2b^3+2ab^4}{(a+b)^2}$ | |

## References


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