A new Slovak mapping projection compatible with GPS

***Adel Alfrehat*[1], *Marcel Mojzeš*[2] *a* *Janka Sabová*[3]**

Nové zobrazenia Slovenska compatibilné s GPS

Matematické transformácie nezabezpečujú exaktné alebo presné výsledky, len čiastočne realizujú kompatibilitu medzi S-JTSK sieťou a ETRS-89. Transformáciou precízne meranej polohy bodov má za následok skreslenie exaktných GPS meraní, preto je vhodnejšie zachovať presnosť meraní s použitím zobrazenia a mapovania, ktoré sú kompatibilné s GPS. Plným využiím dosiahnutej presnosti, získanej z nových geodetických základov Slovenska založených na ETRS-89, ktorý poskytuje vysokú presnosť a je zhodný s medzinárodnými štandardmi, musia byť GPS merania a zobrazenia založené na nových geodetických základoch Slovenska.

Geodeti, inžinieri, navigátori a široká oblasť profesionálnych užívateľov, tak ako aj široká verejnosť stále viac využívajú GPS technológie. Títo užívatelia požadujú možnosť, ako použiť súradnice GPS v mape jednoznačne a rýchlo bez toho, aby museli uvažovať o transformáciách. Na tomto základe je navrhované nové zobrazenie Slovenska – Lambertovo konformné zobrazenie. Referenčný elipsoid tohto zobrazenia je GRS80, a je referenčným elipsoidom ETRS-89 dátumu. Použitím navrhovaného zobrazenia sa dá dosiahnuť plná kompatibilita s GPS meraním. Elipsoidické súradnice určované technikou GPS sa dajú zobrazovať priamo do roviny bez prechodovej transformácie, pri ktorej sa stráca presnosť.

Key words: *Lambert conformal conic projection, GPS, Lambert direct and inverse problems, parameters and advantages of the new Slovak Lambert projection.*

The need for a new Slovak mapping projection

The GPS reference system and the Slovak grid have been derived by different methods, on different datums, within different reference frames and with positions expressed on different coordinate systems. Therefore, GPS positions are not directly compatible with the Slovak grid and they must be transformed in order to relate correctly. The accuracy of the GPS coordinates would be decreased by the datum transformation depending on many factors, the main one being the local deformation of S-JTSK network.

ETRS-89 positions derived using Slovak Kinematic Terrestrial Reference Frame 2001 are three-dimensional, in the form of geographical coordinates. To calculate grid coordinates from the latitude and longitude requires that a map projection is associated with the new geodetic framework, thus providing two-dimensional grid coordinates that can be shown on a map. However, the grid coordinate obtained depends on used ellipsoid and projection parameters ETRS-89 relates to the GRS80 ellipsoid, the Bessel ellipsoid used by the Slovak grid. The new projections need to be associated with the accepted global reference ellipsoid, GRS80, and the associated coordinate system, ETRS-89.

Why the Lambert Conformal Conic Projection?

A candidate map projection for Slovakia is the Lambert Conformal Conic projection for the following reasons:

- The Lambert projection is ideal for mapping states that are narrow north-south, but which extend long distances in an east-west direction as in the Slovak Republic.
- The Lambert projection uses a cone for its developable surface and maintains distance precisions in the east-west direction. Thus it is used in predominantly east-west states. The scale on a Lambert projection varies from north to south but not from east to west.
- The term conformal means that true angular relationships are retained around all points in small regions. This property (conformal) preserves the shape of objects over small regions—a desirable property for surveying and engineering works.
- The area distortion between and near the standard parallels is relatively small. Thus the projection provides an exceptionally good directional and shape relationship for an east-west mid-latitude zone. Consequently, it is used for the air navigation, topographic maps and for meteorological charts in these mid-latitudes.

[1] Ing. Adel Alfrehat, PhD, Dept. of Geodesy and GIS, Univ. Of Košice, Letná 9, 04200 Košice, Slovakia, randogps84@hotmail.com

[2] *doc. Ing. Marcel Mojzeš, PhD,* Dept. of theoritical Geodesy, Univ. Of Bratislava, Radlinskeho 11 813 68 Bratislava, Slovakia, marcel.mojzes@stuba.sk

[3] doc. Dr. Ing. Janka Sabová, Dept. of Geodesy and GIS, Univ. Of Košice, Letná 9, 04200 Košice, Slovakia, *Janka.Sabova@tuke.sk*

(Recenzovaná a revidovaná verzia dodaná 16. 12. 2005)

- The Lambert Conformal Conic projection computations are more easily performed by hand than with others projections.

Parameters and mathematical formulation of the new Slovak projection

The Lambert Conformal Conic Projection, as its name implies, is a projection onto the surface of an imaginary cone as shown in figure (1).

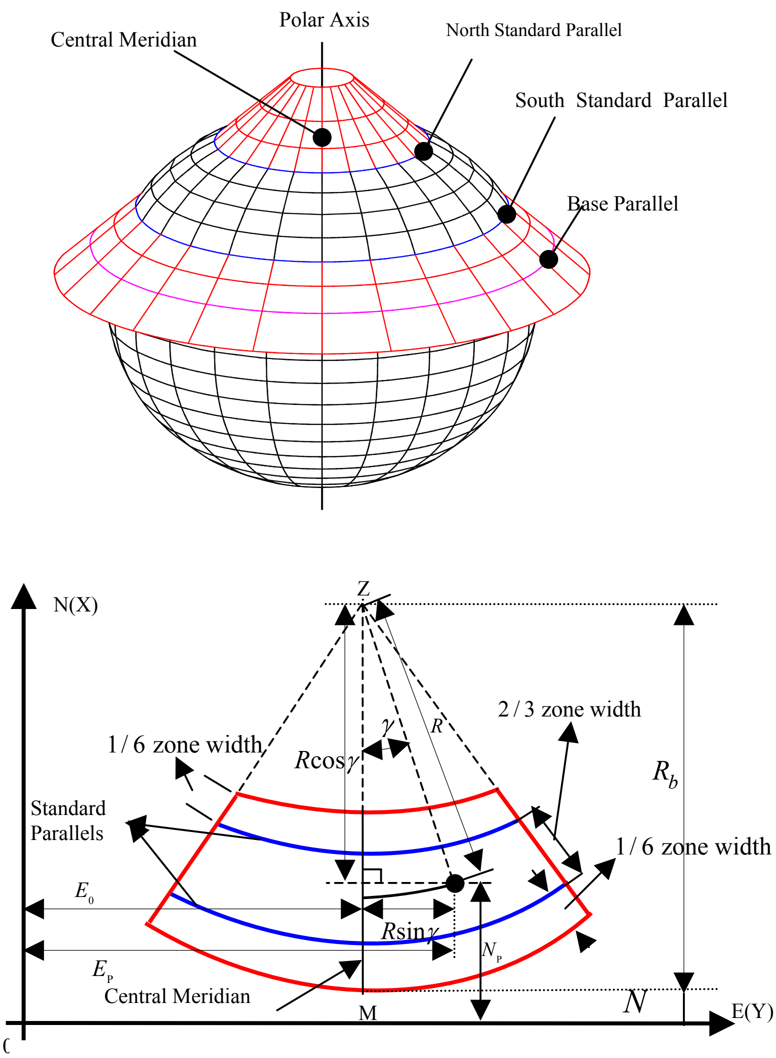

Fig. 1. Portion of the developed cone of a Slovak Lambert projection covering an area of interest

Undistorted standard parallels were selected, so that, the extent of the projected area in the south northern direction was divided with three sections by the rate of 1/6, 2/3 and 1/6. The scale on a Lambert projection varies from north to south but not from east to west. Zone widths in the projection are therefore limited north-south, but not east-west [1]. All necessary parameters for the conversion between the Lambert conformal map projection and the geographic coordinates are identified in the fig. 1.

The standard parallels of a Lambert Conformal Conic projection were placed approximately one-sixth of total zone width from the northern and southern limits. The values of the northern and southern limits were taken from [2]. The approximate positions are

1/6 Total Zone Width: 49°40′-47°40′= 2°/6=20′

North Standard Parallel: 49°40′ – 20′ = 49°20′ N

South Standard Parallel: 47°40′ + 20′ = 48°00′ N

The central meridian is the center of the projection zone. The false easting E_0 is a translation of the Y coordinates of a point, so that all points in the zone are positive. The false northing

N_b is a translation of the X coordinates of a point. R_b is the radius of the projection going from the grid origin to the top of cone along the central meridian to the origin. The standard parallels indicate where the cone intersects with the ellipsoid. Based on figure 2, there is a 1:1 scale along a standard parallel. That is, the length of the line on the ellipsoid is equal to the scale of the line in the grid projection. Lines between the standard parallels have grid lengths that are shorter than their equivalent ellipsoid lengths. All lines outside of them have grid lengths that are longer than equivalent ellipsoid lengths.

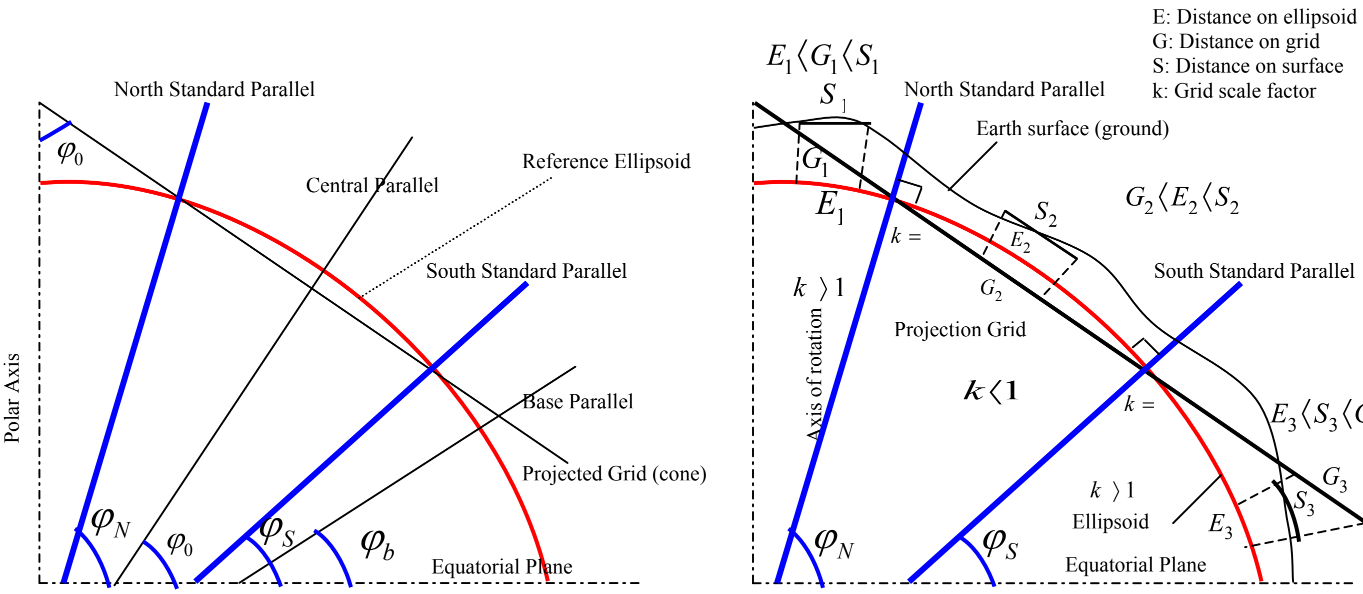

Fig. 2. Lambert conic projection, typical longitudinal section, Lambert Ground, Grid, and Ellipsoid

The grid scale factor k is the ratio between a distance represented on the grid and its corresponding value on the ellipsoid. It is greater than one for regions lying outside the standard parallels, less than one for the region lying between the standard parallels, and exactly one for positions lying on the standards parallels.

The X axis is oriented to the north and the Y axis is oriented to the east [3]. The parameters of Slovak Lambert conformal conic projection are illustrated in fig. 3.

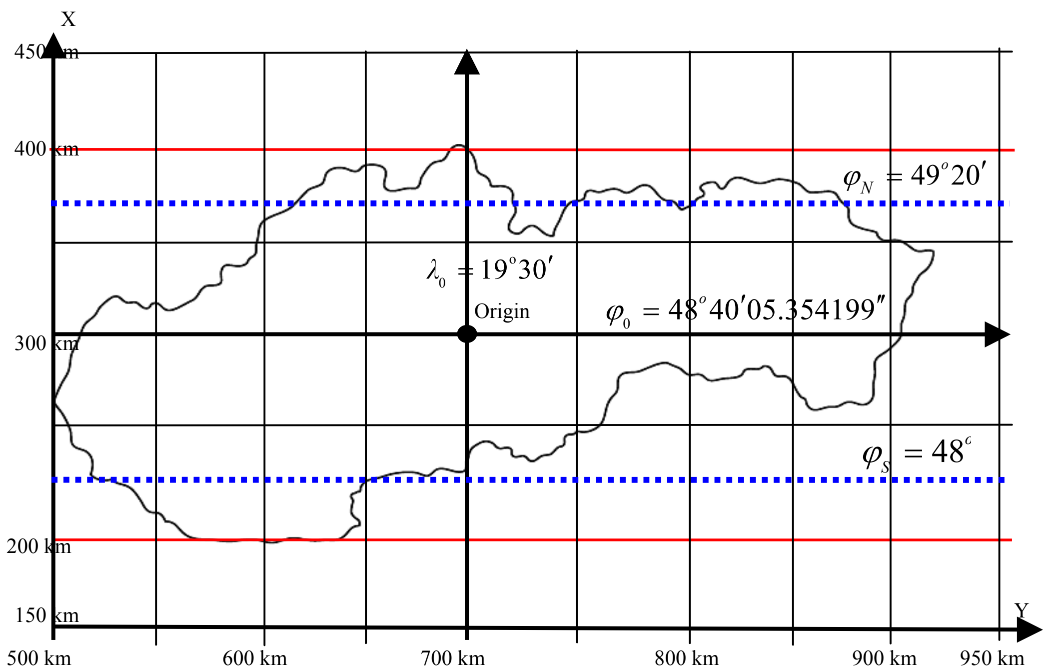

Fig. 3. Parameters of Slovak Lambert conformal conic projection

The values of the suggested parameters for the Slovak Lambert Conformal Conic Projection are in the Tab. 1.

Tab. 1. Parameter values for Slovak Lambert Conformal Conic projection

Parameter	Value
Ellipsoid	GRS 80
a	6378137 m
$1/f$	298.25722210088
Projection Type	Lambert Conformal Conic
φ_0	$48^o 40' 05.354199''$
λ_0	$19^o 30'$
South Standard Parallel φ_S	48^o
North Standard Parallel φ_N	$49^o 20'$
False Easting E_0	500000 m
False Northing N_b	150000 m

Slovak Lambert direct and inverse problems

The two primary computations performed with a map projection are the direct and inverse problems. The direct problem takes geodetic positions (φ,λ) and converts them to equivalent map projection coordinates (Y,X), commonly referred to as easting and northing. The inverse problem takes map coordinates and converts them to geodetic positions.

The formulas for the Lambert Conformal Conic map projection developed for an ellipsoid are shown in the Tab. 2. based on [4], these formulas were modified for the Slovak Lambert Conformal Conic projection definitions and listed in table to using them easily [5]. In these equations, *a* and *f* are the semi-major axis and flattening of the ellipsoid for the selected datum. Since the ETRS-89 datum is based on the Global Reference System of 1980 (GRS 80), and has been officially adopted as a standard coordinate system for the precise GPS surveying by most national mapping agencies in Europe, ellipsoidal coordinates of points measured GPS technique can be directly projected into the Slovak Lambert plane because GPS and a new Slovak projection have the same GRS80 ellipsoid.

The defining map projection parameters include:

k_0, k are the scale factors at the central parallel and at a general point.

φ_0 is the latitude of the (false) grid origin

R_0, R are the mapping radii at latitudes φ_0, φ

q is the isometric latitude

γ is the convergence angle

ε is the base of the natural system of logarithm ε =2,7182818284590452353602875

Tab. 2. Slovak Lambert conformal direct and inverse mapping

Direct Computation	Inverse Computation
Given $a, e, \varphi_0, \lambda_0, \varphi_S, \varphi_N, E_0, N_b,\ and,\ \varphi, \lambda$	Given $a, e, \varphi_0, \lambda_0, \varphi_S, \varphi_N, E_0, N_b,\ and,\ N, E$
Find N, E, k, γ	Find $\varphi\ , \lambda$
$q_0 = \ln\left[tg\left(45^o + \frac{\varphi_0}{2}\right)\left(\frac{1 - e\sin\varphi_0}{1 + e\sin\varphi_0}\right)^{\frac{e}{2}}\right]$,	$N_0 = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_0}}$
$q_S = \ln\left[tg\left(45^o + \frac{\varphi_S}{2}\right)\left(\frac{1 - e\sin\varphi_S}{1 + e\sin\varphi_S}\right)^{\frac{e}{2}}\right]$,	$N_S = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_S}}$
$q_N = \ln\left[tg\left(45^o + \frac{\varphi_N}{2}\right)\left(\frac{1 - e\sin\varphi_N}{1 + e\sin\varphi_N}\right)^{\frac{e}{2}}\right]$,	$N_N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_N}}$
$\sin\varphi_0 = \frac{\ln\left(N_S \cos\varphi_S\right) - \ln\left(N_N \cos\varphi_N\right)}{q_N - q_S}$	
$k_0 = \frac{N_S \cos\varphi_S}{N_0 \cos\varphi_0}\varepsilon^{(q_S - q_0)\sin\varphi_0} = \frac{N_N \cos\varphi_N}{N_0 \cos\varphi_0}\varepsilon^{(q_N - q_0)\sin\varphi_0}$	
Direct Computation	Inverse Computation
$q = \ln\left[tg\left(45^o + \frac{\varphi}{2}\right)\left(\frac{1 - e\sin\varphi}{1 + e\sin\varphi}\right)^{\frac{e}{2}}\right]$ $N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}}$, $\Omega = \varepsilon^{-(q - q_0)\sin\varphi_0}$	$\gamma = arctg\left[\frac{Y - E_0}{k_0 N_0 \cot g\varphi_0 - (X - N_b)}\right]$ $R = \frac{k_0 N_0 \cot g\varphi_0 - (X - N_b)}{\cos\gamma}$ $\Delta q = -\frac{1}{\sin\varphi_0}\ln\left[\frac{R}{k_0 N_0 \cot g\varphi_0}\right]$ $q = q_0 + \Delta q$
Solution	
$\gamma = \sin\varphi_0\left(\lambda - \lambda_0\right)$ $Y = k_0 N_0 \cot g\varphi_0 \sin\gamma\Omega + E_0$ $X = k_0 N_0 \cot g\varphi_0\left(1 - \Omega\cos\gamma\right) + N_b$ $k = \frac{k_0 N_0 \cos\varphi_0}{N\cos\varphi}\Omega$	$\lambda = \lambda_0 + \frac{\gamma}{\sin\varphi_0}$ $\varphi = 2arctg\left[\varepsilon^q\left(\frac{1 + e\sin\varphi}{1 - e\sin\varphi}\right)^{\frac{e}{2}}\right] - 90^o$ (*) $\varphi = 0$ on the right hand side of (*) to Solve for first approximation (iteration)

The advantages of the new Slovak projection

Because the existing S-JTSK and the map projection are not compatible with the modern GPS technique and a new concept in the establishment of new geodetic control networks of Slovakia is not dependent on the existing coordinates‘ system and the map projection, the suggested new projection for the Slovak republic should be considered for the following reasons [5]:

- The adjusted GPS ellipsoidal coordinates determined in ETRS-89 and resulted from the GPS post processing software can be projected directly to a mapping plane using the suggested projection without the loss of accuracy information.

- The suggested parameters of the new projection are exactly defined versus the parameters of the existing Slovak projection where the difference between Greenwich and Ferro meridians has many different values used by users.
- Using the suggested parameters of Slovak Lambert projection, the maximum scale factor of the points of northern and southern limits is 8 cm /km. The scale factor on the territory of the Slovak and Czech Republic is between 0.9999 and 1.00014 and the distance correction using the Krovak projection is between -10 and 14 cm /km [6].
- The scale factor of the point using the suggested projection is a function of only latitude while the scale factor of the point using the Krovak projection is a function of the latitude and longitude.
- The suggested projection and GPS use the same ellipsoid (GRS80) and all GPS measurements can be reduced easily to a new mapping plane in order to combine them with terrestrial measurements.
- Using the suggested projection, GPS, projection and the new geodetic control in Slovakia and the Slovak gravimetric quasigeoid model would have the same reference ellipsoid.
- Most of GPS post processing softwares enable to define the Slovak new projection and then all geodetic calculations can be performed using these softwares and some of them allow to combined GPS and terrestrial data.
- The new Slovak gravimetric quasigeoid model can be added to each GPS post processing softwares database obtaining the normal height of the surveyed points directly.
- All calculations related to the suggested projection are very easy, especially mapping angle.
- All of the transformation problems between GPS and S-JTSK can be avoided using the new suggested projection.
- The suggested projection allows projecting the ellipsoidal coordinates directly on the cone in a normal position. This projection has the normal coordinate system used in geodesy and the uniform size of maps for all scales 50 cm x 50 cm.
- All of the National Spatial Network stations have published coordinates in ETRS-89 and, using the suggested parameters of the new projection, these coordinates can be directly transformed to the Slovak Lambert mapping plane and would have served as a published mapping plane coordinates for the new GPS surveys.

Summary and conclusion

The relation between GPS ellipsoidal coordinates and S-JTSK plane coordinates is not clearly defined by an unique mathematical relation, and GPS coordinates can not be projected directly using the existing Krovak projection. In the absence of distortions in the S-JTSK geodetic network (orientation error on the ellipsoid and variable scale factor), *consistent transformation parameters* between GPS and the Slovak geodetic reference system are impossible to determine over all the country.

The growing numbers of GPS users which have no interest in transformations will therefore be best served by a mapping system which is fully compatible with GPS. This paper has described a new projection of Slovakia and it is an advantage. This new projection minimizes distortions within the new mapping system and realizes ETRS89 coordinates that are substantially different from the existing corresponding Slovak coordinates.

Demonstration example

Fig. 1 shows a network measured by the GPS technique. This network was connected to four control stations (*A*, *B*, *C*, and *D*) from National Spatial Network (ŠPS). Using known coordinates of the control stations in ETRS89 (Tab. 3) and S-JTSK, seven local transformation parameters were computed between ETRS89 and S-JTSK datums to transform the ETRS89 coordinates of the new stations (*F*, *H*, and *G)* to the S-JTSK mapping plane. The transformed S-JTSK coordinates of all stations are shown in Tab. 4.

The GPS coordinates of all stations determined in ETRS89 were projected using the new Slovak projection and the grid Lambert coordinates are listed in Tab. 5.

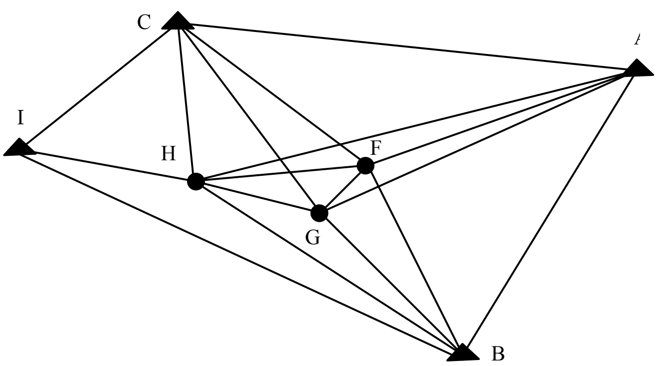

Fig. 4. Configuration of the GPS network

Tab. 3. ETRS89 coordinates of stations

Station	φ [ETRS89]	σ_φ [m]	λ [ETRS89]	σ_λ [m]	h [ETRS89]	σ_h [m]
C	48° 46´ 54.88485″	0,003	21° 12´ 54.99954″	0,002	510,637	0,017
D	48° 44´ 17.69362″	0,003	21° 07´ 17.17172″	0,002	508,788	0,014
B	48° 39´ 40.47797″	0,002	21° 21´ 56.63553″	0,002	313,272	0,014
A	48° 45´ 47.02459″	0,004	21° 27´ 48.46244″	0,003	382,160	0,019
H	48° 43´ 11.91782″	0,003	21° 13´ 16.83682″	0,002	342,733	0,014
F	48° 43´ 52.67649″	0,003	21° 18´ 52.81089″	0,003	348,028	0,017
G	48° 42´ 36.75769″	0,003	21° 17´ 25.93601″	0,003	370,574	0,019

Tab. 4. S-JTSK mapping plane coordinates resulted from the transformation

Station	Y[S – JTSK] [m]	σ_Y [m]	X[S – JTSK] [m]	σ_X [m]	**k**	**k** [cm/km]
C	265605,6085	0,006	1232604,3285	0,006	0,9999171093956	8,289
D	272728,6196	0,007	1237124,3005	0,008	0,9999120608011	8,794
B	255168,1057	0,006	1246519,0935	0,007	0,9999081480785	9,185
A	247478,9615	0,007	1235529,3555	0,008	0,9999178823218	8,212
H	265484,7558	0,005	1239504,6030	0,006	0,9999114572888	8,854
F	258567,9934	0,005	1238566,2667	0,006	0,9999133032879	8,670
G	260449,5494	0,005	1240826,9553	0,006	0,9999112696571	8,873

Tab. 5. Lambert mapping plane coordinates

Station	Y [Lambert] [m]	σ_Y [m]	X [Lambert] [m]	σ_X [m]	**k**	**k** [cm/km]
C	626038,1280	0,003	164066,3180	0,002	0,999934472	6,552
D	619246,9490	0,003	159061,2830	0,002	0,999933252	6,674
B	637419,4220	0,003	150911,7760	0,002	0,999932512	6,748
A	644324,7910	0,002	162410,5260	0,002	0,999933874	6,612
H	626639,1280	0,004	157191,0570	0,003	0,999932913	6,708
F	633473,9250	0,003	158608,7470	0,002	0,999933111	6,688
G	631754,3080	0,003	156222,4790	0,003	0,999932774	6,722

Tab. 6 shows the S-JTSK distances determined from the GPS into S-JTSK plane (e.g., computed from S-JTSK transformed coordinates), the grid distances computed from the Slovak Lambert plane coordinates and the differences between them in meters.

Tab. 6. Comparison between distances computed from coordinates inthe S-JTSK and the Lambert mapping plane

Distance	S-JTSK Distances (m)	Lambert Distances (m)	Difference (m)
A F	11497,3674	11497,6005	0,2330
B G	7764,9266	7765,1196	0,1931
D H	7624,9199	7625,0938	0,1740
C F	9223,4881	9223,6738	0,1857
C H	6901,3327	6901,4792	0,1464
F G	2941,2524	2941,3190	0,0666
G H	5205,9504	5206,0743	0,1240
F H	6980,1201	6980,2790	0,1589
A B	13412,5792	13412,8808	0,3017
A C	18361,1305	18361,4730	0,3425
C D	8436,0793	8436,2603	0,1810

For the network of fig. 1, the S-JTSK mapping plane coordinates were obtained from the transformation and the accuracy of these coordinates were in average 0,007 m and 0,006 m in the *X* and *Y* axis (tab. 4 and compare this accuracy with the accuracy level of the obtained GPS ellipsoidal coordinates of new points, which was achieved by tying the GPS survey to the ŠPS network) while the accuracy of the projected coordinates using the Slovak Lambert projection was in average 0,002 m and 0,003 m in the *X* and the *Y* axis (tab. 5).

The maximum scale factor for point *B* in the case of the Krovak projection was 9,185 cm/km and in the case of the Slovak Lambert projection was 6,75 cm/km. The maximum differences between the computed grid distances from S-JTSK and the Lambert grid coordinates is 34 cm for 18361 m (tab. 6).

References

[1] Wolf, P., R., et al.: Elementary surveying, an introduction to geomatic, *3 Ed, 2002*.

[2] Valko, J., et al.: Aplikácia kartografických zobrazení pre Slovensko, *Zborník z 11. Kartografickej konferencie, Bratislava 1995*.

[3] Mojzeš, M.: Nové Zobrazenia Slovenska, *Seminář s mezinárodní účastí k 105. Výročí založení České vysoké školy technické v Brně, BRNO, 2004*.

[4] Leick, A.: GPS Satellite Surveying. *New York, Chicester, Toronto, Brisbane, Singapore: John Wiley and Sons Inc, 1995*.

[5] Alfrehat, A.: Spracovanie GPS meraní v S-JTSK a novom zobrazení Slovenska, *PhD thesis, Technical University of Kosice, Institute of geodesy and GIS, June 2005*.

[6] Abelovič, J., et al.: Meranie v geodetických sieťach. *Alfa, Bratislava 1990*.