Partial eigenvalue assignment problem of linear control systems using orthogonality relations

***Mohamed A. Ramadan*[1] and *Ehab A. El - Sayed*[2]**

Problém priradenia parciálnych charakteristických hodnôt lineárnych riadiacich systémov použitím ortogonálnych závislostí

The partial eigenvalue assignment is the problem of reassigning a part of the open-loop spectrum of a linear system by a feedback control, leaving the rest of the spectrum invariant. In the paper, we propose a novel solution to the partial eigenvalue assignment problem of linear control system using orthogonality relations between eigenvectors of the state matrix A. Our solution can be implemented with only a partial knowledge of the spectrum and the corresponding eigenvectors of the linear system. We show that, the number of eigenvalues and eigenvectors that need to remain unchanged will not be affected by the feedback. We prove this, the feedback vector must be of real form. A numerical example is given to illustrate the proposed method.

Key words: *linear control systems, orthogonality relations, eigenvectors, matrix*

1. Introduction

Consider the linear-time invariant single-input control system

$$\dot{x} = Ax + bu \tag{1.1}$$

where A is a $n \times n$ real non-symmetric constant matrix, b is a n-column vector and u is a scalar function of t. It is known [2,13] that, if the pair (A,b) is controllable we can choose a feedback vector f such that the closed loop $\dot{x} = (A - bf^T)x$ has a desired set of eigenvalues. This closed loop system can be achieved by applying a state-feedback control $u = -f^T x$. Such problem is as known *the single-input eigenvalue assignment problem*, which is of great importance in many applications in the control theory; see for example [1,7].

This single-input eigenvalue assignment problem is studied through the conventional numerical methods (e.g. the implicit QR and real Schur methods as in [8,9]). However, in control problems of large and sparse structure it is often desirable to modify only a few of the eigenvalues of the open-loop system and leave the rest unchanged. Such problem is called *the partial pole assignment problem*.

Mathematically given the matrix A whose spectrum $\{\lambda_1, \lambda_2, \cdots, \lambda_m, \lambda_{m+1}, \cdots, \lambda_n\}$ and set $\{\mu_1, \mu_2, \cdots, \mu_m\}$, closed under complex conjugation then the partial eigenvalue assignment problem for the linear control system (1.1) requires us to find the feedback vector f such that spectrum of $(A - bf^T)$ is $\{\mu_1, \mu_2, \cdots, \mu_m, \lambda_{m+1}, \cdots, \lambda_n\}$. Through this paper we assume that the pair (A,b) is controllable.

Many authors [6,10,12] introduced projection algorithms for solving the partial eigenvalue assignment problem of the linear control system (1.1). where the technique proposed by Datta and Saad [6] first solves a Sylvester- type equation $AX - XH = GC$ (H is an upper Hessenberg matrix constructed using Arnoldi's algorithm) such that the spectrum of H is $\{\mu_1, \mu_2, \cdots, \mu_m\}$, the matrix G is of the special form $G = (0,0,\cdots,0,b)$ and X is an orthonormal matrix obtained from the orthonormal matrix V constructed from Arnoldi's algorithm where $V^T AV = H$, by multiplying X with a chosen scalar. The Sylvester-type equation is then rewritten in an equivalent form so that the matrix X will be the basis of the invariant subspace of A, and the required feedback vector $f = Xe_m$, where $e_m = (0,0,\cdots,0,1)^T$, so that

[1] *Mohamed A. Ramadan*, Department of Mathematics, Faculty of Science, Menoufia University, Egypt

[2] *Ehab A. El – Sayed*, Department of Science and Mathematics, Faculty of Petroleum Engineering, Suez Canal University, Egypt, ***ehab_math@yahoo.com***

(Recenzovaná a revidovaná verzia dodaná 2. 2. 2006)

the spectrum of $\left(A - bf^T\right)$ will contain $\{\mu_1, \mu_2, \cdots, \mu_m\}$. While the techniques in the methods [10,12] are similar and depend mainly on the notion of computing an orthonormal basis, using Arnoldi's algorithm as in [10] or using the partial Schur QR factorization as in [12], of an invariant subspace associated with the eigenvalues that are needed to be assigned and then apply one of the known and suitable methods to solve a resulting associated eigenvalue assignment problem of a smaller dimension. There are numerical difficulties associated with these projection algorithms.

In our present method, the solution of the partial eigenvalue assignment problem is obtained using orthogonality relations between eigenvectors of a matrix A [3,11].

The solution of the partial eigenvalue assignment problem can be obtained by using only a partial knowledge of eigenvalues and eigenvectors of the matrix A. This solution allows us to work directly with the matrix A without using any type of projection

2. Orthogonality Relations between the Eigenvectors of a Matrix

In this section, we introduce orthogonality relations between eigenvectors for the eigenvalue problem

$$Ax = \lambda x. \tag{2.1}$$

Definition 1

Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue and a non–zero column vector x is called the right eigenvector corresponding to λ if

$$Ax = \lambda x \tag{2.3}$$

Definition 2

Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue and a non –zero column vector y satisfying

$$y^H A = \lambda y^H \tag{2.4}$$

is called the left eigenvector corresponding to λ, where y^H is the conjugate transpose of the vector y.

Definition 3

The pairs (λ, x) and (λ, y) are called, respectively, right and left eigenpairs of A.

Datta and D.R. Sarkissian [3,11] stated the orthogonality relations of the eigenvectors of a matrix A as given in the theorem 1 below.

Theorem 1 [3,11] (Orthogonality of the Eigenvectors of a Matrix A)

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of matrix A and let X and Y be, respectively, the right and the left eigenvector matrices. Assume that $\{\lambda_1, \cdots, \lambda_m\} \cap \{\lambda_{m+1}, \cdots, \lambda_n\} = \Phi$. The partition $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$, where $X_1 = (x_1, \cdots, x_m)$, $X_2 = (x_{m+1}, \cdots, x_n)$, $Y_1 = (y_1, \cdots, y_m)$ and $Y_2 = (y_{m+1}, \cdots, y_n)$.

Then

$$Y_1^H X_2 = 0 \tag{2.5}$$

and

$$Y_1^H A X_2 = 0 \tag{2.6}$$

If, in addition, A is real symmetric, then

$$X_1^T X_2 = 0 \text{ and } X_1^T A X_2 = 0 \tag{2.7}$$

In the next section, we solve the partial eigenvalue assignment problem of (1.1) by using the orthogonality relation (2.6).

3. Partial Eigenvalue Assignment Problem

Suppose that $A \in R^{n \times n}$ be a non-symmetric matrix. Let the eigenvalue problem $Ax_i = \lambda_i x_i$ $i = 1,2,\cdots,n$, be written in the matrix form:

$$AX - X\Lambda = 0 \tag{3.1}$$

where $X = (x_1, x_2, \cdots, x_n) \in C^{n \times n}$, $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \in C^{n \times n}$ and λ_i are distinct.

Given m complex numbers $\{\mu_1, \mu_2, \cdots, \mu_m\}$ closed under complex conjugation, $m \le n$ and a vector $b \in R^n$, we are required to find $f \in R^n$ such that the matrix $(A - bf^T)$ has a spectrum.

$$\{\mu_1, \mu_2, \cdots, \mu_m, \lambda_{m+1}, \cdots, \lambda_n\} \tag{3.2}$$

Let us partition the $n \times n$ right eigenvector matrix X, the $n \times n$ left eigenvector matrix Y^H and $n \times n$ eigenvalues matrix Λ as follows:

$$X = (X_1 \quad X_2),\ Y^H = \begin{pmatrix} Y_1^H \\ Y_2^H \end{pmatrix},\ \Lambda = diag(\Lambda_1, \Lambda_2)$$

where $X_1 = (x_1, \cdots, x_m)$; $X_2 = (x_{m+1}, \cdots, x_n)$, $Y_1 = (y_1, \cdots, y_m)$ and $Y_2 = (y_{m+1}, \cdots, y_n)$ with $\Lambda_1 = diag(\lambda_1, \cdots, \lambda_m)$ and $\Lambda_2 = diag(\lambda_{m+1}, \cdots, \lambda_n)$.

Theorem 2

If $\{\lambda_1, \cdots, \lambda_m\} \cap \{\lambda_{m+1}, \cdots, \lambda_n\} = \Phi$ and the feedback vector f defined by

$$f^T = \beta Y_1^H A \tag{3.3}$$

then for any choice of β, the last $n - m$ eigenvalues $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$ of the matrix $(A - bf^T)$ are the same as those of the matrix A.

Proof

Let (X, Λ) be the eigenvector-eigenvalue matrix pair of the matrix A, then

$$AX - X\Lambda = 0.$$

Our goal is to prove that:

$$(A - bf^T)X_2 - X_2\Lambda_2 = 0. \tag{3.4}$$

By substituting $f^T = \beta Y_1^H A$ in the left hand side (3.4), we obtain

$$(A - bf^T)X_2 - X_2\Lambda_2 = AX_2 - X_2\Lambda_2 - b\beta(Y_1^H AX_2).$$

Since $AX_2 - X_2\Lambda_2 = 0$ and $Y_1^H AX_2 = 0$ from the theorem 1, thus

$$(A - bf^T)X_2 - X_2\Lambda_2 = 0.$$

The theorem is then proved.

3.1 Choosing β

In order to use the theorem 2 to solve the partial eigenvalue assignment problem, we need to choose β that moves $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ of A to $\{\mu_1, \mu_2, \cdots, \mu_m\}$ in $(A - bf^T)$, if it is possible. If there is such β, then exists an eigenvector matrix $Z \in C^{n \times m}$;

$$Z = (z_1, z_2, \cdots z_m), \quad z_j \ne 0, \quad j = 1,2,\cdots,m.$$

and matrix $D = diag(\mu_1, \mu_2, \cdots, \mu_m)$ such that

$$(A - bf^T)Z - ZD = 0 \tag{3.5}$$

Substituting in (3.5) for $f^T = \beta Y_1^H A$, we have

$$AZ - b\beta Y_1^H AZ - ZD = 0$$

and then

$$AZ - ZD = b\beta Y_1^H AZ = b\beta W^H = bc^H \tag{3.6}$$

where $W^H = Y_1^H AZ$ and $c = W\beta^H$ is a vector that will depend on the scaling chosen for the eigenvectors in Z. To obtain Z, we choose the vector c as $c = (1,1,\cdots,1)^T$. Then equation (3.6) becomes

$$AZ - ZD = b(1,1, \quad \ldots, \quad 1).$$

We can solve for each of the eigenvectors z_j using the equations

$$(A - \mu_j I)z_j = b \qquad j = 1,2,\cdots,m \tag{3.7}$$

So, we obtain the eigenvectors of Z, and hence compute the matrix W from $W^H = Y_1^H AZ$. We solve the $m \times m$ square linear system

$$W\beta^H = (1,1,\cdots,1)^T \tag{3.8}$$

for β^H, and hence we can determine the vector f.

4. Explicit Expression for β

In the next theorem, we obtain an explicit expression for β using only a partial knowledge of eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and the associated left eigenvectors $\{y_1, \cdots, y_m\}$ of the matrix A. Our proof to this theorem is similar to the proof of the theorem introduced in [4,5] for the symmetric definite quadratic pencil case.

Theorem 3

Suppose $Ax_i = \lambda_i x_i$, $1 \le i \le n$ has the form (3.1) and f is chosen as in (3.3) with the components β_j of β as

$$\beta_j = \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j} \prod_{\substack{i=1 \\ i \ne j}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i}, \quad j = 1,2,\cdots,m, \tag{4.1}$$

then the matrix $(A - bf^T)$ has the spectrum $\{\mu_1, \mu_2, \cdots, \mu_m, \lambda_{m+1}, \cdots, \lambda_n\}$ and its first m eigenvectors can be scaled to satisfy $(A - \mu_j I)z_j = b \qquad j = 1,2,\cdots,m$.

Proof.

We need only to show

$$\Phi_k(\beta) = [(A - bf^T) - \mu_k I]z_k = 0, \quad k = 1,2,\cdots,m. \tag{4.2}$$

where

$$(A - \mu_k I)z_k = b. \tag{4.3}$$

Substituting the expression $f^T = \beta Y_1^H A$ in $\Phi_k(\beta)$ gives

$$\Phi_k(\beta) = [(A - \mu_j I) - b\beta Y_1^H A]z_k.$$

Then, from (4.3), we have

$$\Phi_k(\beta) = b - [b\beta Y_1^H A]z_k$$

Now, substituting for β_j using (4.1) gives

$$\Phi_k(\beta) = b - \left[b\left(\sum_{j=1}^{m} \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} y_j^H A \right) \right] z_k$$

$$= b - \left[b\left(\sum_{j=1}^{m} \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j - \lambda_k} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} y_j^H (\lambda_j - \mu_k) \right) \frac{A}{\lambda_j} \right] z_k ,$$

then

$$= b - \left[b\left(\sum_{j=1}^{m} \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j - \lambda_k} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} \right) y_j^H (\lambda_j - \mu_k) \frac{A}{\lambda_j} \right] z_k .$$

The j-th column of (2.4) can be written as

$$y_j^H I = y_j^H \frac{A}{\lambda_j}, \quad \lambda_j \neq 0.$$

Hence for any choice of $1 \leq k, j \leq m$,

$$y_j^H (A - \mu_k I) = y_j^H \left(A - \mu_k \left(\frac{A}{\lambda_j} \right) \right),$$

then

$$y_j^H (A - \mu_k I) = y_j^H (\lambda_j - \mu_k) \left(\frac{A}{\lambda_j} \right). \qquad (4.4)$$

Substituting (4.4) into the last expression of $\Phi_k(\beta)$ gives

$$\Phi_k(\beta) = b - \left[b\left(\sum_{j=1}^{m} \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j - \lambda_k} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} \right) y_j^H (A - \mu_k I) \right] z_k ,$$

and using (4.3) we get

$$\Phi_k(\beta) = b - b \sum_{j=1}^{m} \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j - \lambda_k} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} y_j^H b ,$$

$$= b - b \sum_{j=1}^{m} \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j - \lambda_k} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} \left(b^T \bar{y}_j \right)^T .$$

Canceling the common term, we get

$$\Phi_k(\beta) = b - b \sum_{j=1}^{m} \frac{\lambda_j - \mu_j}{\lambda_j - \lambda_k} \prod_{\substack{i=1 \\ i \neq j,k}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i} ,$$

$$\Phi_k(\beta) = b \left(1 - \sum_{j=1}^{m} \frac{\prod_{\substack{i=1 \\ i \neq k}}^{m} \lambda_j - \mu_i}{\prod_{\substack{i=1 \\ i \neq j}}^{m} \lambda_j - \lambda_i} \right).$$

In [5] it is proved that

$$\sum_{j=1}^{m} \frac{\prod_{\substack{i=1\\ i\neq k}}^{m} \lambda_j - \mu_i}{\prod_{\substack{i=1\\ i\neq j}}^{m} \lambda_j - \lambda_i} = 1, \quad k = 1,2,\cdots,m, \tag{4.5}$$

for any sets of $\{\lambda_i\}_{i=1}^{m}$ and $\{\mu_i\}_{i=1}^{m}$ in which the λ_i are distinct, and thus $\Phi_k(\beta)$ vanishes, as required. This completes the proof of the theorem.

From the expression (4.1) it is clear that sufficient conditions for the existence of β, and consequently for a solution to the partial eigenvalue assignment problem to exist:

a, No λ_j, $j = 1,2,\cdots,m$ vanishes,

b, The $\{\lambda_i\}_{i=1}^{m}$ are distinct,

c, The vector b must be not orthogonal to $\bar{y}_j$, $j = 1,2,\cdots,m.$

The above discussion leads us to formulate the following algorithm for our solution of the partial eigenvalue assignment problem.

Algorithm The single–input partial eigenvalue assignment algorithm

Inputs: A is an $n \times n$ real non-symmetric constant matrix, b is an *n*-vector and $D = diag(\mu_1, \mu_2, \cdots, \mu_m)$, closed under a complex conjugation.

Assumption: The numbers $\mu_1, \cdots, \mu_m; \lambda_1, \cdots, \lambda_n$ are all distinct and closed under a complex conjugation, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the matrix A.

Output: The feedback vector f such that the spectrum of the matrix $(A - bf^T)$ is $\{\mu_1, \cdots, \mu_m; \lambda_{m+1}, \cdots, \lambda_n\}$, where $\lambda_{m+1}, , \cdots, \lambda_n$ are the last $n - m$ eigenvalues of A.

Step 1. Obtain the first m eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ of the matrix A that need to be reassigned and the corresponding left eigenvectors $y_1, y_2, \cdots, y_m$.

Step 2. Compute the explicit expression for β where its components are given as:

$$\beta_j = \frac{1}{b^T \bar{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j} \prod_{\substack{i=1\\ i\neq j}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i}, \quad j = 1,2,\cdots,m.$$

Step 3. Form

$$f^T = \beta Y_1^H A.$$

5. Real Form of a Feedback Vector f .

In this section we prove that the feedback vector f must be real, if all $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and $\{\mu_1, \mu_2, \cdots, \mu_m\}$ are closed under a complex conjugation.

Theorem 4

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and $\{\mu_1, \mu_2, \cdots, \mu_m\}$ be two disjoint sets of complex numbers, closed under a complex conjugation and let $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ are a part of the eigenvalues of the matrix A with $A \in R^{n\times n}$. We assume that the non–zero column vectors $\{y_1, y_2, \cdots y_m\}$ be such that $\bar{y}_k = y_j$. Then

$\beta_k = \overline{\beta}_j$ whenever $\overline{\lambda}_j = \lambda_k$ and $\overline{\mu}_j = \mu_k$ $j = 1,2,\cdots,m, \quad j \neq k$ where β_j is a component of β as in (4.1).

Proof

We take

$$\beta_j = \frac{1}{b^T \overline{y}_j} \frac{\lambda_j - \mu_j}{\lambda_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{\lambda_j - \mu_i}{\lambda_j - \lambda_i}, \quad j = 1,2,\cdots,m. \tag{5.1}$$

then

$$\overline{\beta}_j = \frac{1}{b^T y_j} \frac{\overline{\lambda}_j - \overline{\mu}_j}{\overline{\lambda}_j} \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{\overline{\lambda}_j - \overline{\mu}_i}{\overline{\lambda}_j - \overline{\lambda}_i}, \quad j = 1,2,\cdots,m. \tag{5.2}$$

Since $\overline{\lambda}_j = \lambda_k$, $\overline{\mu}_j = \mu_k$ and $\overline{y}_k = y_j$ then

$$\overline{\beta}_j = \frac{1}{b^T \overline{y}_k} \frac{\lambda_k - \mu_k}{\lambda_k} \prod_{\substack{i=1 \\ i \neq k}}^{m} \frac{\lambda_k - \mu_i}{\lambda_k - \lambda_i} = \beta_k \quad k = 1,2,\cdots,m, \tag{5.3}$$

and hence $\overline{\beta}_j = \beta_k \quad j,k = 1,2,\cdots,m. \quad j \neq k.$ The theorem is then proved.

Now, in the following theorem we describe how to transform a complex conjugate set of β and the set of left eigenvectors Y_1^H to the real ones. This will be required to obtain the real feedback vector f .

Theorem 5

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ be a set of complex numbers, closed under a complex conjugation, $\{\beta_1, \beta_2, \cdots, \beta_m\}$ and let vectors $\{y_1, y_2, \cdots y_m\}$ be such that $\overline{y}_k = y_j$ and $\beta_k = \overline{\beta}_j$ whenever $\overline{\lambda}_j = \lambda_k$, $j = 1,2,\cdots,m, \quad j \neq k$.

Then

I- There exists a nonsingular matrix $T \in C^{m \times m}$ such that

$$T^{-1} = T^H, \quad \beta_R = \beta T^H, \quad Y_{1R}^H = T Y_1^H, \tag{5.4}$$

where $\beta = \{\beta_1, \beta_2, \cdots, \beta_m\} \in C^{1 \times m}$ and $Y_1 = \{y_1, y_2, \cdots y_m\} \in C^{n \times m}$ and both β_R and Y_{1R}^H are real matrices.

II-There exists a real feedback vector f such that

$$f^T = \beta_R Y_{1R}^H A \tag{5.5}$$

Proof

I- Define

$$T = \begin{bmatrix} S & 0 & 0 & \cdots & 0 \\ 0 & S & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & S & 0 \\ 0 & 0 & \cdots & 0 & S \end{bmatrix}, \tag{5.6}$$

where

$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and $S^H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$. Then the matrix S satisfies:

$S^H S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus the matrix T is nonsingular where $T^{-1} = T^H$.

Rewrite $\beta = \{\beta_1, \beta_2, \cdots, \beta_m\}$ in the other form $\beta = \{\beta_{1,2}, \beta_{3,4}, \cdots, \beta_{j,,j+1}, \cdots, \beta_{m-1,m}\}$ where $\beta_{j,j+1} = \{\beta_j, \beta_{j+1}\}$. Since $\beta_k = \overline{\beta}_j$, we assume that $\beta_{j+1} = \overline{\beta}_j$ and $\beta_j = a + ib$, then $\beta_{j,j+1} = \{a+ib, a-ib\}$

and

$$\beta_{j,j+1}S^H = \{a+ib, a-ib\}S^H = \sqrt{2}\{a,b\}$$
$$\beta T^H = \{\beta_{1,2}S^H, \beta_{3,4}S^H, \cdots, \beta_{j,j+1}S^H, \cdots, \beta_{m-1,m}S^H\} = \beta_R.$$

Set $Y_1 = \{y_1, y_2, \cdots y_m\}$ in the block form $Y_1 = \{y_{1,2}, y_{3,4}, \cdots, y_{j,,j+1}, \cdots, y_{m-1,m}\}$ where $y_{j,j+1} = \{y_j, y_{j+1}\}$. Since $\overline{y}_k = y_j$, we assume that $y_{j+1} = \overline{y}_j$ and $y_j = c_j + id_j$, where both $c_j = [c_{1j}, c_{2j}, \cdots, c_{nj},]^T$ and $d_j = [d_{1j}, d_{2j}, \cdots, d_{nj},]^T$ are columns then

$$y_{j,j+1} = \{c_j + id_j, c_j - id_j\}.$$
$$y_{j,j+1}S^H = \{c_j + id_j, c_j - id_j\}S^H = \sqrt{2}\{c_j, d_j\}$$
$$Y_1T^H = \{y_{1,2}S^H, y_{3,4}S^H, \cdots, y_{j,j+1}S^H, \cdots, y_{m-1,m}S^H\} = Y_{1R}. \qquad (5.7)$$

By using the transpose conjugate of (5.7), we obtain $Y_{1R}^H = TY_1^H$

II- Now, we show that the feedback vector f must be real. In the section 3, we showed that the spectrum of the matrix $(A - bf^T)$ is $\{\mu_1, \mu_2, \cdots, \mu_m, \lambda_{m+1}, \cdots, \lambda_n\}$ such that $f^T = \beta Y_1^H A$.

Since

$$\beta_R = \beta T^H \quad and \quad Y_{1R}^H = TY_1^H,$$

then

$$f^T = \beta Y_1^H A = \beta T^H T Y_1^H A = \beta_R Y_{1R}^H A$$

where both β_R and Y_{1R}^H are real matrices. Then, there is a real feedback vector

$$f^T = \beta_R Y_{1R}^H A$$

such that the spectrum of the matrix $(A - bf^T)$ is $\{\mu_1, \mu_2, \cdots, \mu_m, \lambda_{m+1}, \cdots, \lambda_n\}$. The theorem is proved.

Remark:

Clearly, if all $\{\lambda_j\}_{j=1}^m$ are real, then Y_1^H is real as well. If, in addition, all $\{\mu_j\}_{j=1}^m$ are real, then both β and f are also real.

6. A numerical Example

We choose a randomly generated matrix A (size 8) and a randomly vector b as follows:

$$A = \begin{bmatrix} 0.8214 & 0.9355 & 0.1389 & 0.4451 & 0.8381 & 0.3046 & 0.3784 & 0.8180 \\ 0.4447 & 0.9169 & 0.2028 & 0.9318 & 0.0196 & 0.1897 & 0.8600 & 0.6602 \\ 0.6154 & 0.4103 & 0.1987 & 0.4660 & 0.6813 & 0.1934 & 0.8537 & 0.3420 \\ 0.7919 & 0.8936 & 0.6038 & 0.4186 & 0.3795 & 0.6822 & 0.5936 & 0.2897 \\ 0.9218 & 0.0579 & 0.2722 & 0.8462 & 0.8313 & 0.3028 & 0.4966 & 0.3412 \\ 0.7382 & 0.3529 & 0.1988 & 0.5252 & 0.5028 & 0.5417 & 0.8998 & 0.5341 \\ 0.1763 & 0.8132 & 0.0153 & 0.2026 & 0.7095 & 0.1509 & 0.8216 & 0.7271 \\ 0.4057 & 0.0099 & 0.7468 & 0.6721 & 0.4289 & 0.6979 & 0.6449 & 0.3093 \end{bmatrix}$$

.

$$b = \begin{bmatrix} 0.9501 \\ 0.2311 \\ 0.6068 \\ 0.4860 \\ 0.8913 \\ 0.7621 \\ 0.4565 \\ 0.0185 \end{bmatrix}$$

The matrix A has eigenvalues shown in Tab. 1.

Eigenvalues of A
-0.2035 + 0.6192i
-0.2035 - 0.6192i
0.6823 + 0.0225i
0.6823 - 0.0225i
4.1633
0.3323
0.0862
-0.6793

Now, we assign the first $m = 2$ eigenvalues λ_1, λ_2 to the conjugate pair $\mu_{1,2} = -3 \pm i$. Using the explicit formula (4.1) gives

$$\beta = [-40.0110 - 6.1360i \quad -40.0110 + 6.1360i],$$

from which we compute the feedback vector f, in view of (3.3) as:

$$f = \begin{bmatrix} -11.2870 \\ -8.8118 \\ 12.5617 \\ 20.7621 \\ -8.3749 \\ 2.4033 \\ -12.7117 \\ 4.1311 \end{bmatrix}.$$

As expected, in the theorem 5, f must be real.

The eigenvalues of the matrices A and $\left(A - bf^T\right)$ are shown in Tab. 2.

Eigenvalues of A	**Eigenvalues of** $\left(A - bf^T\right)$
-0.2035 + 0.6192i	-3.0000 + 1.0000i
-0.2035 - 0.6192i	-3.0000 - 1.0000i
0.6823 + 0.0225i	0.6823 + 0.0225i
0.6823 - 0.0225i	0.6823 - 0.0225i
4.1633	4.1633
0.3323	0.3323
0.0862	0.0862
-0.6793	-0.6793

7. Conclusion

In this paper, we derived an explicit solution to the partial eigenvalue problem by using one of the orthogonality relations between the eigenvectors for the linear pencil $Ax - \lambda x = 0$. We need only a partial knowledge of the spectrum (and the associated left eigenvectors) of the matrix A. These eigenvalues and eigenvectors are required to be reassigned. We proved that the solution (feedback vector f) for this problem is in the real form.

8. References

[1] Barnett, S.: Introduction to Mathematical Control Theory, *Clarendon Press, Oxford, 1975*.

[2] Datta, B., N.: An Algorithm to Assign Eigenvalues in a Hessenberg Matrix: Single Input Case, *IEEE Trans. Auto. Control, Vol. AC-32, No. 5, pp. 414 – 417, 1987*.

[3] Datta, B., N.: Numerical Linear Algebra and Applications, *Brook/Cole Publishing Co., Pacific Grove, California, 1998.*

[4] Datta, B., N.: Recent developments on nonmodal and partial modal approaches for control of vibration, *Applied Numerical Mathematics 30, pp 41-52, 1999*.

[5] Datta, B., N., Elhay, S., Y., Ram, M.: Ortogonality and Partial Pole Assignment for the Symmetric Definite Quadratic Pencil, *Lin. Alg. Appl. 257, 29-48, 1997.*

[6] Datta, B., N., Saad, Y.: Arnoldi methods for large Sylvester-like observer matrix equation and partial pole placement, *Linear Algebra Appl. 154/156, pp. 225-244, 1991*.

[7] Kailath, T.: Linear Systems, *Prentice – Hall, Englewood Cliffs, NJ, 1980.*

[8] Mimins, G., S., Paige., C., C.: An algorithm for pole assignment of time Invariant linear systems, *internet. J. Control, Vol. 35, No. 2, pp. 341–354, 1982*.

[9] Patel, R., V., Misra, P.: Numerical algorithms for eigenvalue assignment by state feedback, *Proc. IEEE, 72, pp. 1755–1764, 1984.*

[10] Ramadan, M., A.: On the projection methods for partial eigenvalue assignment in large systems, *Bull. FAC. SCI., Assiut Univ., 24 (2 – C), pp.103-112, 1995.*

[11] Sarkissian, D., R.: Theory and Computations of Partial Eigenvalue and Eigenstructure Assignment Problems in Matrix Second-Order and Distributed-Parameter systems, *Ph.D. thesis, Dept. of Mathematics, Northern Illinois, University, IL, 2001.*

[12] Saad, Y. Projection and deflation methods for partial pole assignment in linear state feedback, *IEEE Trans. Auto. Control, Vol. 33, No.3, pp. 290-296, 1988.*

[13] Wonham, W., M.: Linear Multivariable Control: A Geometric Approach, *Springer-Verlag New York. 1979.*