Method for simulation of the fractional order chaotic systems

Ivo Petráš [1]

Abstract

This paper deals with the method of simulation of fractional order chaotic systems. We present a brief survey of the existing fractional order chaotic systems. These systems are described by three fractional differential equations where order of derivatives is a non-integer, arbitrary order. The total order of chaotic system is less than three. We present an approach where we demonstrate by an illustrative example the method for deriving the model of such a kind of fractional order chaotic system and method for its simulation. This example is well known chaotic system, the so called Chua's oscillator. We have demonstrated the real measurements and the simulation in the Matlab/Simulink as well.

Key words: *fractional calculus, Chua's system, fractional-order chaotic system, chaos, strange attractor.*

Introduction

A number of applications where fractional calculus has been used rapidly grows. These mathematical phenomena allow to describe a real object more accurate than the classical integer methods. The real objects are generally fractional (Oustaloup, 1995; Podlubny, 1999; Westerlund, 2002) however, for many of them the fractionality is very low. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy transmission line (Wang, 1987) or diffusion of heat into a semi-infinite solid, where the heat flow is equal to the half-derivative of temperature according to time (Podlubny, 1999).

The main reason for using the integer-order models was the absence of solution methods for fractional differential equations. We have to identify and describe the real object by the fractional order models. The first advantage is that we have more degrees of freedom in the model. The second advantage is that we have a "memory" in the model. Fractional-order systems have an unlimited memory, being integer-order systems cases in which the memory is limited.

It is well-known that a chaos cannot occur in continuous systems of total order less than three. This assertion is based on the usual concepts of order, such as the number of states in a system or the total number of separate differentiations or integrations in the system. The model of system can be rearranged to three single differential equations, where the equations contain the non-integer (fractional) order derivative. The total order of system is changed from 3 to the sum of each particular order. To put this fact into context, we can consider the fractional-order dynamical model of the system. Hartley et al. consider the fractional-order Chua's system; in the work by Arena et al., 1998, (the fractional order cellular neural network was considered, in the work by Gao et al., 2005 the fractional Duffing's systems was presented. Other fractional order chaotic systems were described in the many others works (e.g. Chunguang et al. 2004, Deng et al. 2005, Lu 2005, Nimmo et al., 1999, etc.). In all these cases the chaos was exhibited in a system with the total order less than three.

Fractional calculus

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695.

The fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator (1), where *a* and *t* are the limits of the operation. The continuous differential operator is defined as

$$ {}_aD_t^r = \begin{cases} \dfrac{d^r}{dt^r} & \Re(r) > 0, \\ 1 & \Re(r) = 0, \\ \int_a^t (d\tau)^{-r} & \Re(r) < 0. \end{cases} \qquad (1) $$

[1] *Doc. Ing. Ivo Petráš, PhD.,* Department of Applied Informatics and Process Control, ÚRVP - BERG - Technical University of Košice, B. Němcovej 3, Košice, Slovak Republic, ivo.petras@tuke.sk
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The two definitions used for the general fractional differintegral are the Grünwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition (Oldham 1974, Podlubny 1999). The GL is given here:

$$ {}_aD_t^r f(t) = \lim_{h \to 0} h^{-r} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{r}{j} f(t - jh), \tag{2}$$

where [.] means the integer part. The RL definition is given as

$$ {}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} d\tau, \tag{3}$$

for (n - 1 < r < n) and where Γ(.) is the *Gamma* function.

Fractional-order chaotic system

For an illustration we have chosen the classical Chua's oscillator which can be realized by electrical elements according to the scheme shown in Fig. 1, which is a simple electronic circuit (Kennedy, 1992) that exhibits a nonlinear dynamical phenomenon such as the bifurcation and chaos.

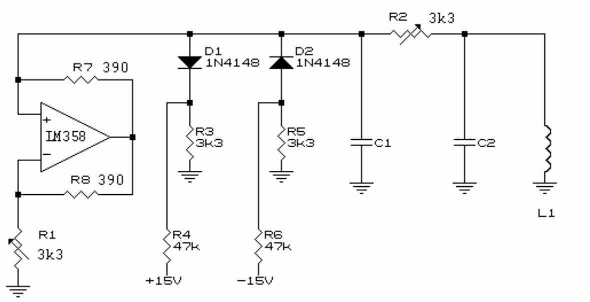

Fig. 1. Circuit of Chua's oscillator.

This circuit behavior can be described by three differential equations. Westerlund et al. in 1994 proposed a new linear capacitor model. It is based on Curie's empirical law of 1889 which states that the current through a capacitor is

$$ i(t) = \frac{U_0}{h_1 t^m}, \tag{4}$$

where h_1 and m are constant, U_0 is the *dc* voltage applied at $t = 0$, and $0 < m < 1$.

For a general input voltage $u(t)$ the current is

$$ i(t) = C \frac{d^m u(t)}{dt^m}, \tag{5}$$

where C is capacitance of the capacitor. It is related to the kind of dielectrics. Another constant m (order) is related to the losses of the capacitor. Westerlund provided in his work the table of various capacitor dielectric with an appropriate constant m obtained by measurements.

Westerlund in his work (Westerlund, 2002) also described the behavior of a real inductor. For a general current in the inductor, the voltage is

$$ u(t) = L \frac{d^m i(t)}{dt^m}, \tag{6}$$

where L is inductance of the inductor and the constant m is related to the "proximity effect".

Applying the Kirchhoff laws and relation (5) and (6) into the circuit depicted in Fig. 1, we get the following mathematical model of Chua's system (Petráš, 2006):

$$
\begin{aligned}
C_1 \frac{d^{q_1} v_1}{dt^{q_1}} &= G(v_2 - v_1) - f(v_1), \\
C_2 \frac{d^{q_2} v_2}{dt^{q_2}} &= G(v_1 - v_2) + i, \\
L_1 \frac{d^{q_3} i}{dt^{q_3}} &= -v_2,
\end{aligned} \tag{7}
$$

where v_1 is voltage on the capacitor C_1, v_2 is voltage on the capacitor C_2, i is current in the inductor L_1, $G = 1/R_2$, q_1 is the real order of the capacitor C_1, q_2 is the real order of the capacitor C_2, q_3 is the real order of the inductor L_1, and $f(v_1)$ is the piecewise linear v - i characteristic of the nonlinear Chua's diode, which can be described as

$$f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)(|v_1 + E| - |v_1 - E|) \qquad (8)$$

with E being the breakpoint voltage of a diode, and G_a<0 and G_b<0 being some appropriate constants (the slope of the piecewise linear resistance). Chua's diode (8) – negative impedance converter was realized by operating the amplifier LM 358 and the resistors R_1, R_7 and R_8 ($R_7 = R_8$).

Illustrative example

Experimental measurements

For an experimental verification of Chua's system depicted in Fig. 1 and described by equations (7) and (8) were chosen the following values of electrical elements:

$$C_1 = 4.71nF,\ C_2 = 48nF,\ L_1 = 4.64mH,\ R_1 = 897\Omega,\ R_2 = 998k\Omega,\ R_7 = R_8 = 390\Omega \qquad (9)$$

We used a ceramic capacitors C_1 and C_2 with the real order $q_1 = q_2 = 0.98$ and we assume the real order of inductor $q_3 = 0.94$ (Westerlund, 2002). Total order of the system is $q = 2.90$.

Assuming the three-segment piecewise-linear voltage transfer characteristic of negative impedance converter (8), we have the slopes

$$G_a = -\frac{1}{R_1},\ \text{(for } R_7 = R_8\text{)},\ G_b = -\frac{1}{R_7} \qquad (10)$$

The breakpoint of the non-linear characteristic (8) is $E \approx 9$ [V] (experimentally found). The resistors R_3, R_4, R_5, R_6, and the diodes D_1 and D_2 generate the positive and negative half of the non-linearity.

In Fig. 2 is depicted a photo of the oscilloscope screen. It is a real measurement of voltages v_1 - v_2 of the circuit presented in Fig. 1, for the electrical components (9) and parameters (10), projected onto the v_1 - v_2 plane.

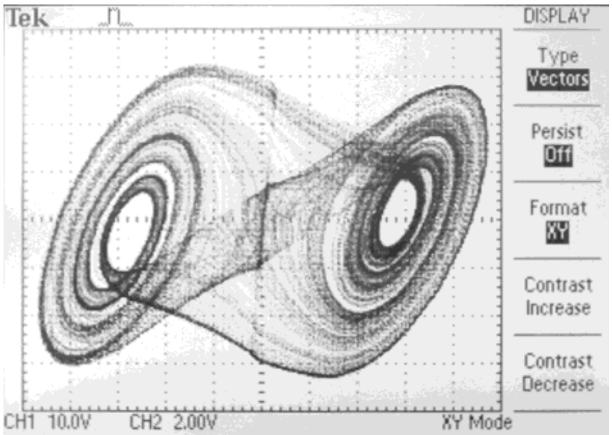

Fig. 2. Strange attractor of Chua's system (7).

The result shown in Fig. 2 is the double-scroll attractor of fractional order Chua's system described by the equations (7) and (8). The measurements were done by the digital oscilloscope Tektronix TDS1002, 60 Mhz.

Simulation in Matlab/Simulink

For a simulation in the Matlab/Simulink environment we will use a dimensionless form of the fractional order Chua's system which can be described by the following three differential equations ($x \equiv v_1/E$, $y \equiv v_2/E$, $z \equiv i/EG$, $A \equiv C_2/C_1$, $B \equiv C_2/(L_1G^2)$, $m_0 \equiv G_a/G$, $m_1 \equiv G_b/G$):

$$\begin{aligned} {}_0D_t^{q_1} x(t) &= A(y(t) - x(t) - f(x)), \\ {}_0D_t^{q_2} y(t) &= x(t) - y(t) + z(t), \\ {}_0D_t^{q_3} z(t) &= -By(t), \end{aligned} \qquad (11)$$

where the nonlinear function $f(x)$ is described as

$$f(x) = m_1 x(t) + 0.5(m_0 - m_1) \times (|x(t)+1| - |x(t)-1|) \qquad (12)$$

We have used the Matlab/Simulink approach. The block diagram of the simulation is depicted in Fig. 3. For the simulation of the fractional derivative (integral) we used a Simulink block `nid` created by Duarte Valerio with the combination of the classical integrator using the property of commutation by two operators. In the mentioned block for the fractional order operator, we used a continuous fraction expansion method of the generating function. The order of such an approximation in the form of a rational function was n = 10 (Vinagre et al., 2003).

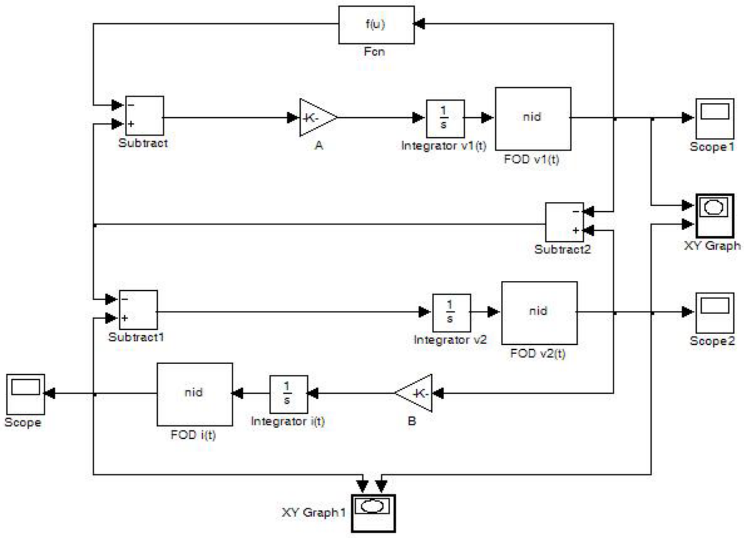

Fig. 3. Block diagram of Chua's system in the Matlab/Simulink.

In Fig. 4 is depicted the simulation result obtained by the numerical simulation in the Matlab/Simulink for the following values of parameters: A = 10.19, B = 10.30, $q_1 = q_2$ = 0.98 and q_3= 0.94, for the initial conditions: (x, y, z) = [0.6, 0.1, -0.6] and for the slopes of Chua's diode (12) characteristics: m_0 = -1.11 and m_1 = -0.86.

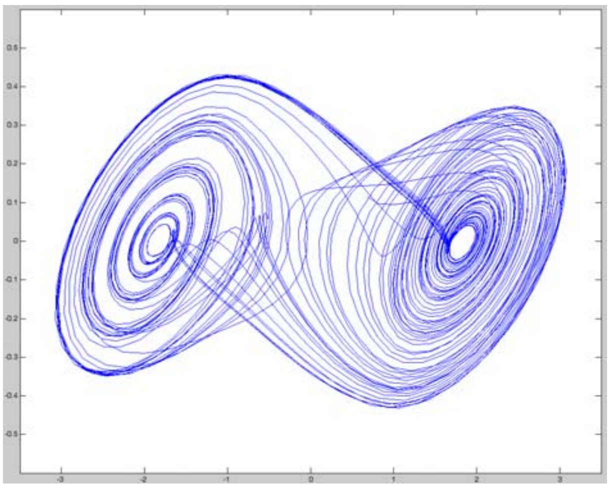

Fig. 4. Simulation result (x vs. y) of Chua's system (11).

Conclusion

We have considered an example of chaotic fractional-order Chua's circuit, which exhibits a chaotic behavior with the total order less than three. As has been demonstrated, the idea of fractional calculus requires one to reconsider dynamic system concepts that are often taken for granted. So, changing the order of a system from integer to real, we also move from a three-dimensional system to the infinite dimension.

The conclusion of this work confirms the conclusions of the works (Hartley et al., 1995; Lu, 2005; Podlubny, 1999; Westerlund; 2002), that there is a need to refine the notion of the order of a system which can not be considered only by the total number of differentiation. For fractional-order differential equations the number of terms is more important than the order of differentiation.

The fractional-order model for chaotic Chua's system was directly derived because electrical elements used in the circuit are not ideal. As was mentioned in the work by Westerlund (2002), the real electrical elements (e.g. capacitors, inductors, etc.) have a fractional order and should be described by fractional-order models.

As has been demonstrated, the idea of fractional calculus requires to reconsider the dynamical system concepts. Some of them have been noted in this article and also in the work by Petráš at al. (2006). We also presented an approach to a simulation of the fractional order chaotic system in the Matlab/Simulink environment. The results obtained from the simulation are comparable with the results from a real measurement.

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