Electronic realization of the fractional-order systems

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Elektronická realizácia systému neceločíselného rádu.

This article is devoted to the electronic (analogue) realization of the fractional-order systems – controllers or controlled objects whose we earlier used, identified, and analyzed as a mathematical models only – namely a fractional-order differential equation, and solved numerically using a method based on the truncated version of the Grunwald - Letnikov formula for fractional derivative. The electronic realization of the fractional derivative is based on the continued fraction expansion of the rational approximation of the fractional differentiator from which we obtained the values of the resistors and capacitors of the electronic circuit. Along with the mathematical description are presented also simulation and measurement results.

Key words*: fractional-order dynamical system, fractional-order differentiator, continued fraction expansion, domino ladder, symmetrical domino ladder*

Introduction

Fractional-order calculus (FOCA) is about 300 years old topic and it means a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$, $\alpha \in R$, where a and t are the limits and α is the order of the operation. The theory of fractional-order (FO) derivative was developed mainly in the 19th century. In the last decades there has been, besides the theoretical research of FO derivatives and integrals (Oldham et al, 1974ô Samko et al, 1987; Podlubný, 1994; 1999; Chen et al., 2002; Lubich, 1986; Diethelm, 1997; etc.) a growing number of applications of FO calculus in many different areas such as, for example, long electrical lines, electrochemical processes, dielectric polarization, colored noise, viscoelastic materials, chaos and of course in control theory as well (Manabe, 1961; Outstaloup, 1988; 1995; Axtell et al., 1990; Dorčák, 1994; Vinagre et al., 2000; Petráš et al., 2002a; Chen et al., 2002). This is a confirmation of the statement that real objects are generally fractional-order, however, for many of them the fractionality is very low.

As we mentioned in the abstract, the mainspring for the creation of an electronic realization of the FO controlled object was the ambition to create a real, working object, whose mathematical model (Dorčák, 1994) we have already been using for more than ten years. The first explicit non-iterative numerical methods for simulation of such FO systems was developed (Dorčák, 1994) and the first comparison with the corresponding analytical methods derived from analytical solutions published in (Podlubný, 1994) was made. This numerical method is based on the truncated version of the Grunwald-Letnikov formula for FO derivative. An example of experimental design of a FO PD$^\delta$ controller, with comparison of its dynamic properties, was made with this model too. It follows from these results that the design of the regulator of an FO system with integer-order model as the best approximation of the FO system is inadequate and with a change of the system or regulator parameters can lead to system instability. It is better for research and educational purposes to have a real FO controlled object which we can identify (Dorčák et al., 1996; Podlubný, 1998) and which we can use to compare different types of integer- and fractional-order controllers (Vinagre et al., 2001; Petráš et al., 2001; Dorčák et al., 2006; etc.).

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Methods of description and solution of fractional order systems

In the time domain we can describe FO systems by an FO differential equation or by systems of FO differential equations. Very frequent, as a model of the controlled system in control theory, is the following three-term FO differential equation

$$a_2\, y^{(\alpha)}(t) + a_1\, y^{(\beta)}(t) + a_0\, y(t) = u(t), \tag{1}$$

where α, β are generally real numbers, a_2, a_1, a_0 are arbitrary constants. For one kind of our final desired FO controlled objects they have the following values $\alpha = 2.2$, $\beta = 0.9$, $a_2 = 0.8$, $a_1 = 0.5$, $a_0 = 1$. In the case $a_2 = 0$ we have two-term FO differential equation. Analytical solution of FO differential equations [Podlubný 1994, 1999] is rather complicated. More convenient are numerical solutions. In general, there are two discretization methods: direct discretization and indirect discretization. We will consider the direct methods – first is a power series expansion (PSE), second is continued fraction expansion (CFE). The PSE method is based on the Grunwald-Letnikov (GL) definition of the differ-integral operator

$${}_aD_t^r f(t) = \lim_{T\to 0}\frac{1}{T^r}\sum_{j=0}^{\left[\frac{t-a}{T}\right]} b_j^{(r)} f(t-jT) = \lim_{T\to 0}\frac{1}{T^r}\Delta_T^r\, f(t), \tag{2}$$

where $[x]$ means the integer part of x, $\Delta f(t)$ is the generalized finite difference of order r and time step T, $b_j^{(r)}$ are binomial coefficients $b_0^{(r)} = 1, b_j^{(r)} = \left(1-(1+(\pm r))/j\right) b_{j-1}^{(r)}$. Using GL definition (2) we can derive (Dorčák, 1994) explicit equation for the numerical solution of equation (1) and corresponding analytical solution derived from analytical solutions (Podlubný, 1994):

$$y_k = \frac{u_k - a_1 T^{-\beta}\sum_{i=1}^{k} b_j y_{k-j}}{a_1 T^{-\beta} + a_0},\quad k = 1,2,\ldots \text{ for } \beta = (0,1\rangle,\quad k = 2,3,\ldots \text{ for } \beta = (1,2\rangle \tag{3}$$

$$y(t) = \frac{1}{a_1} t^{\beta} E_{\beta,\beta+1}\left(-\frac{a_0}{a_1} t^{\beta}\right),\quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{4}$$

Of course, for special cases β=1 or 2 we have also the following classical analytical solutions of two-term FOE: $y(t)=1/a_0(1-\exp(-a_0/a_1 t))$, $y(t)=1/a_0(1-\cos((a_0/a_1)^{1/2}t))$.

The second discretization method is based on continued fraction expansion (CFE). As generating function we can use a simple Euler's backward difference, the Tustin rule, the Simpson rule or the Al-Alaoui operator, which is mixed scheme of Euler's rule and the Tustin trapezoidal rule. Muir recursion applied to the Tustin generating function (GF) and CFE applied to the Tustin or Al-Alaoui GF leads to the following approximations of the FO differentiator/integrator:

$$D^{\pm\delta}(z) = \left(\frac{K_1}{K_2 T}\right)^{\pm\delta} \lim_{n\to\infty}\left(\frac{A_n(z^{-1},+\delta)}{A_n(z^{-1},-\delta)}\right) = \left(\frac{K_1}{K_2 T}\right)^{\pm\delta}\frac{P_n(z^{-1},+\delta)}{Q_n(z^{-1},-\delta)} \tag{5}$$

$$D^{\pm\delta}(z) = \left(\frac{K_1}{K_2 T}\right)^{\pm\delta} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}/K_2}\right)^{\pm\delta}\right\}_{p,q} = \left(\frac{K_1}{K_2 T}\right)^{\pm\delta}\frac{P_p(z^{-1})}{Q_q(z^{-1})} \tag{5}$$

where T is the time step or sample period, δ is the order of differentiator or integrator, K_1 and K_2 are the constants according to the GF (K_1=2, K_2=1 for the Tustin GF, K_1=8, K_2=7 for the Al-Alaoui GF) (Dorčák, 1994), P and Q are polynomials of degrees p and q, respectively, in the variable z^{-1} ($n=p=q$ for Muir recursion).

Realization of fractional-order systems

The fractional-order integro-differential operator we can design and built on the principle clearly visible on Fig. 1. The basic element of this circuit is the appropriate impedance Z – so called fractance in the first stage, which along with resistance R_i defines the fractional-order of the operator. The function of the second stage is to determine the required gain of the whole FO operator or to compensate the gain of the first stage (Fig. 1).

Above mentioned continuous fraction expansions (CFE) (Chen et al., 2002; Petráš et al., 2002a; Vinagre et al., 2001; Dorčák, 1994; etc.), are also used as a tool for designing circuits for analogue realizations of FO dynamical systems (Kvasil et al., 1967; Petráš et al., 2001; Podlubný et al., 2002; Petráš et al., 2002b; Dorčák 1994; etc).

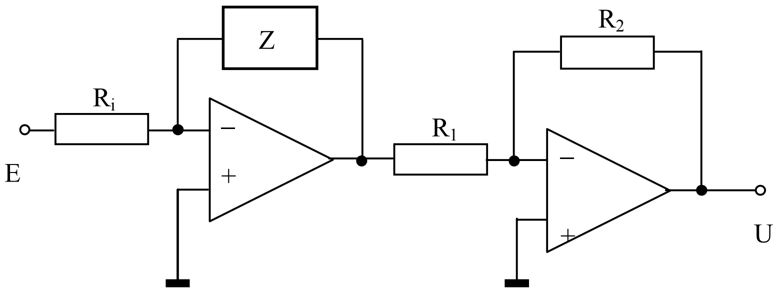

Fig. 1. Diagram of the electronic realization of the fractional-order operator.

The resulting discrete transfer function of e.g. Tustin's rule, can be expressed as

$$D^{\pm\beta}(z) = \frac{Y(z)}{U(z)} = \left(\frac{2}{T}\right)^{\pm\beta} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm\beta}\right\}_{p,q} = \left(\frac{2}{T}\right)^{\pm r} \frac{P_p^{\beta}(z^{-1})}{Q_q^{\beta}(z^{-1})}, \tag{6}$$

where $Y(z)$ and $U(z)$ are Z transforms of the output and input sequences $y(nT)$, $u(nT)$. P and Q are polynomials of degrees p and q, respectively, in the variable z^{-1}.

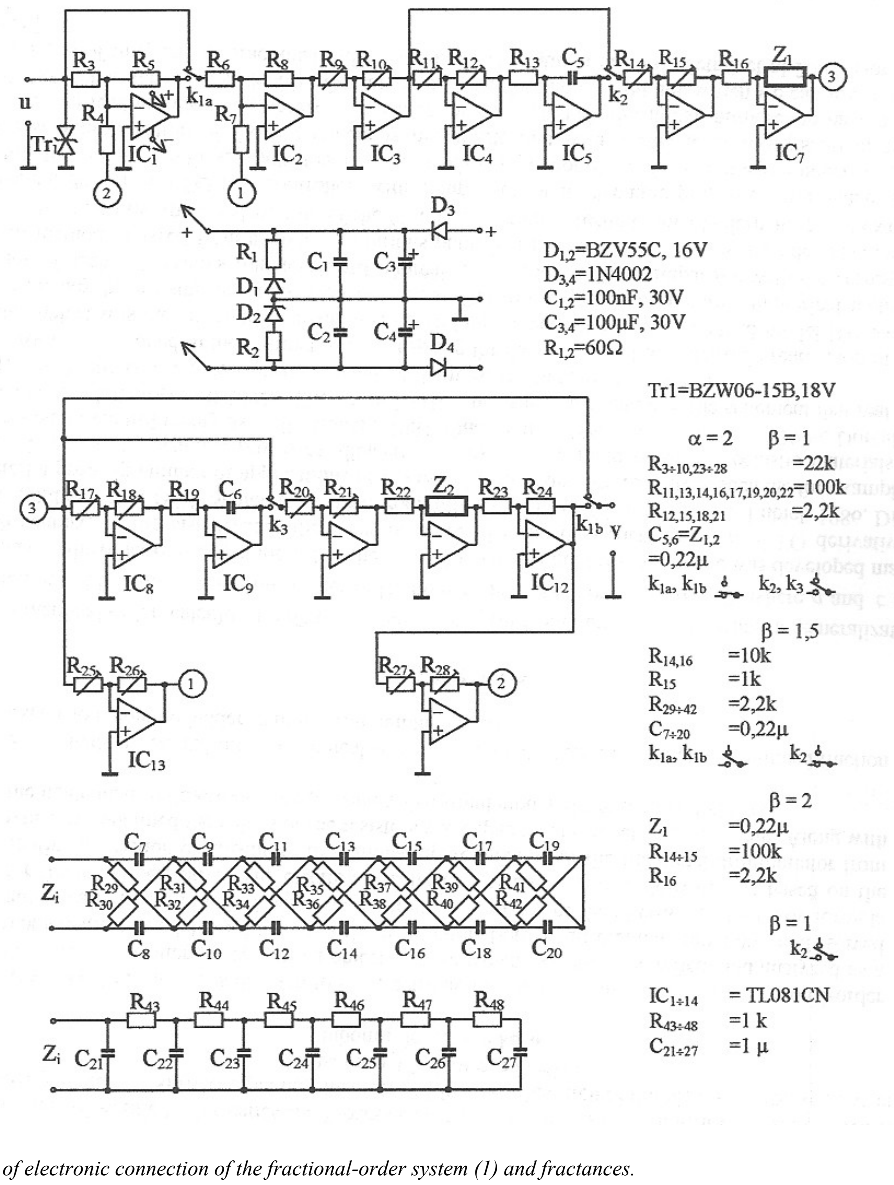

Fig. 2. Diagram of electronic connection of the fractional-order system (1) and fractances.

The basic types of impedances Z for linear circuits for simulation of FO integro-differential operator are domino ladder, transmission line circuit (Fig. 2) or symmetrical domino ladder and tree circuits. The use of CFE allows the unification of the design of the two types of circuits for analogue simulation of FO systems, namely the domino ladder circuit and tree circuit. Truncated CFE gives a rational approximation, which does not require any further transformation. The values of the electronic elements, which are necessary for building a fractance, are then determined from the coefficients of the obtained finite continued fraction. If all coefficients are positive, then the fractance can be made of classical elements (resistors and capacitors) (Petráš et al., 2002b).

Using the FO integrator and the method of successive iteration of the highest derivative (α) of the output variable $y^{(\alpha)}(t)$ expressed from the equation (1) we can design the diagram depictured in Fig. 2.

While in mathematical methods described by (3), (4) we have not so strong limitation for the order of the FO derivatives, for electronic realization of the FO derivatives we have the following limitation for the derivative order $\beta \in (0, 1\rangle$. Therefore we must combine integer- and fractional-order differentiators for the electronic realization (Fig. 2) of the possible systems described by FO differential equation (1).

We can simulate then with circuit depicted in Fig. 2, Fig. 5 some different types of integer- and fractional-order systems for α and $\beta \in (0, 2\rangle$. Firstly we compare in this article the non-typical case of two-term FO differential equation with β=1.5, $a_1 = 1$, $a_0 = 1$. We have also tested special cases with β=1 and 2. In Fig. 3 and in Fig. 4 are depicted unit step responses and corresponding Bode plots of the FDEs. Very interesting is the fact, that the two terms FO system (1) for $a_2 = 0$ and $\beta \in (1, 2)$ has an oscillatory behaviour, because of the characteristic equation has a couple of complex-conjugate roots.

Behaviours of the fractional-order system (1) for special case of $\beta = 1$, $a_2 = 0$ and e.g. $\alpha = 0$, $a_1 = 0.5$, $a_0 = 1$ measured on the analogue electronic realization looks of course like the behaviours of the classical first order system. Behaviour of the FO integro-differential operator (integrator) alone (Fig. 1) to the unit step response for β=0.5 and gain 0.58 is depictured in Fig. 6.

The steps response of our well-known system (1) for $\alpha = 2.2$, $\beta = 0.9$, $a_2 = 0.8$, $a_1 = 0.5$, $a_0 = 1$ are depictured in Fig. 7 and corresponding Bode plots in Fig. 8.

The described method works for arbitrary orders of the FO operator, but the circuit elements with computed values are not usually available. Suitable for realization with standard resistors and capacitors is operator of the orders $\beta = 0.5$, $\beta = 1.9$, etc. The very important task for the future is to investigate the methods for design of the arbitrary real orders.

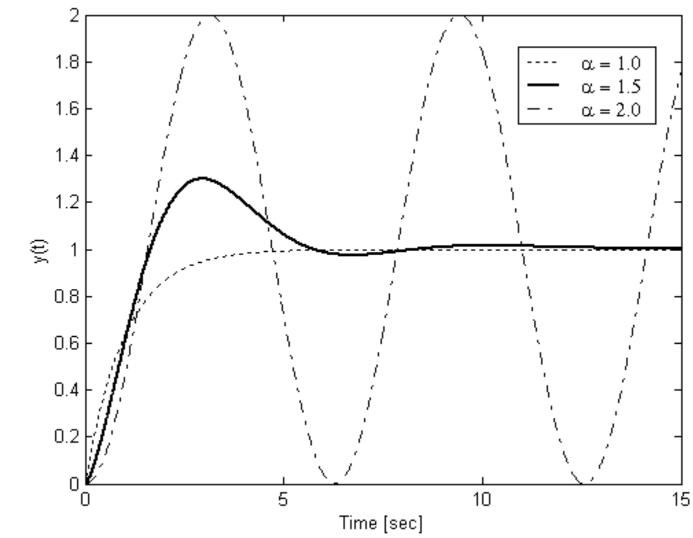

Fig. 3. Unit step responses.

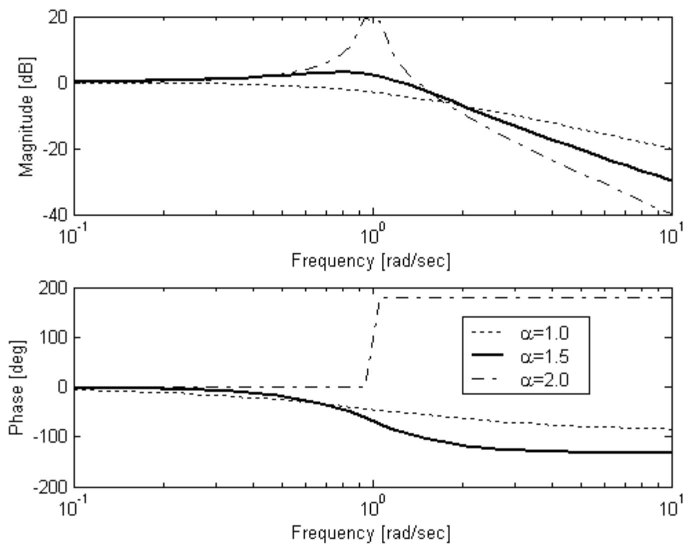

Fig. 4. Bode plots of the FDEs.

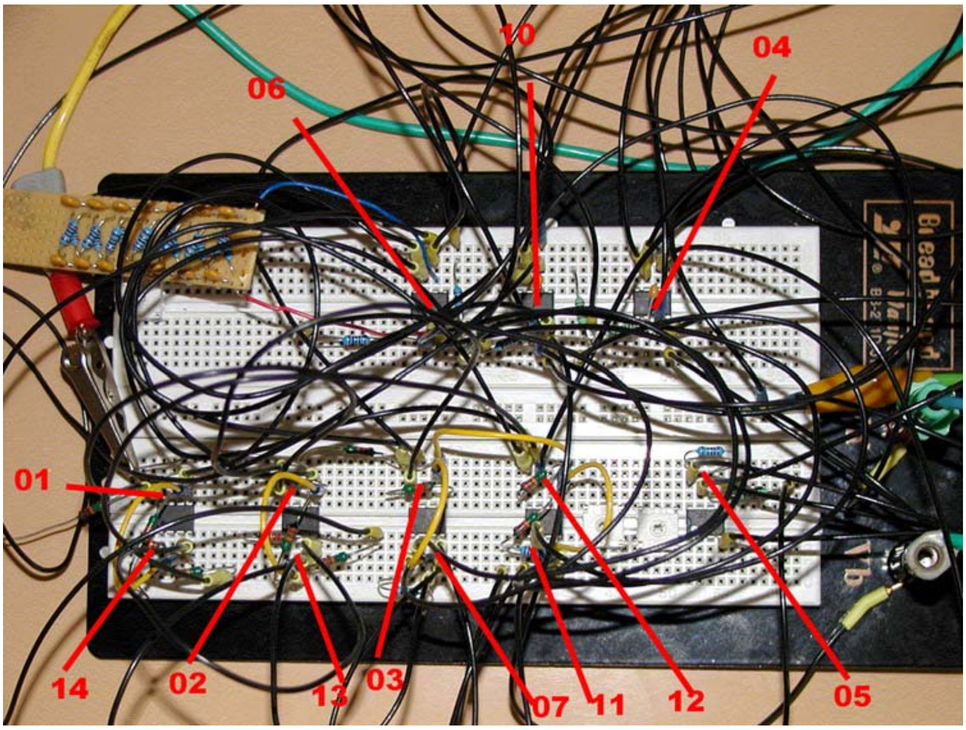

Fig. 5. Photo of the analogue electronic realization of the FO controlled object.

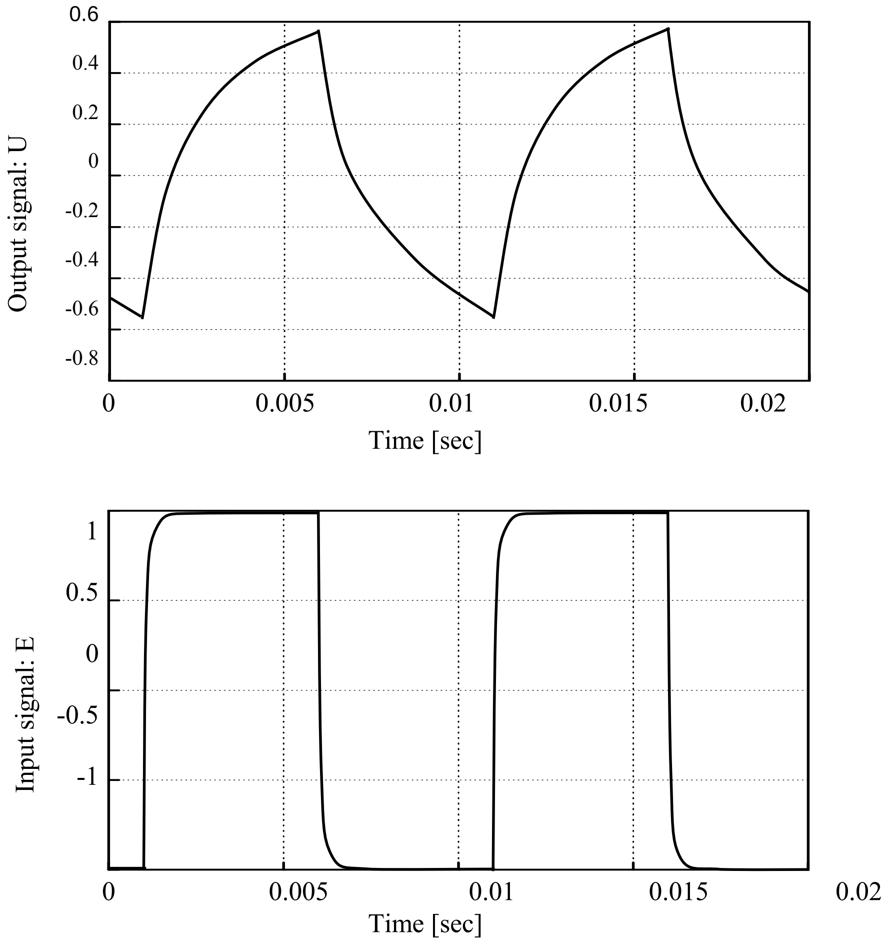

Fig. 6. Unit step responses of the fractional-order integrator.

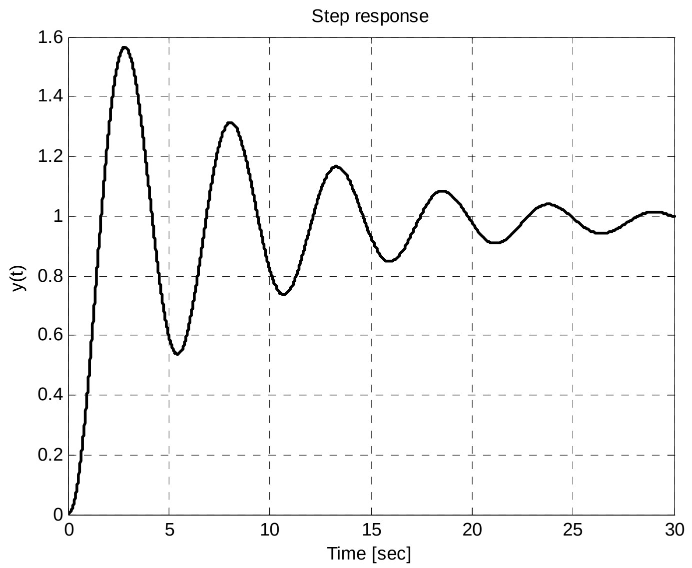

Fig. 7. Unit step response.

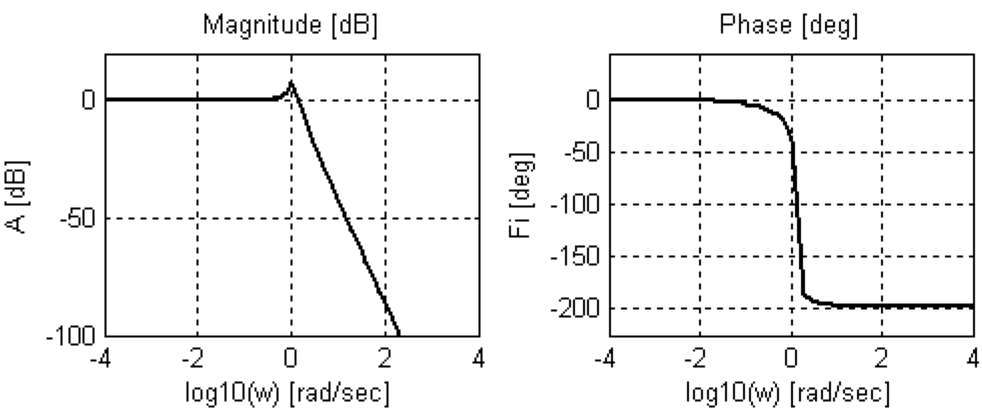

Fig. 8. Bode plots.

Conclusions

In this article we have presented the electronic (analogue) realization of the fractional-order systems, controllers or controlled objects. The electronic realization of the fractional derivative is based on the continued fraction expansion of the rational approximation of the fractional differentiator from which we obtained the values of the resistors and capacitors of the electronic circuit.

The presented method works for arbitrary orders of the FO operator, but the circuit elements with computed values are not usually realizable with standard resistors and capacitors. The very important task for the future is to investigate the methods for design of the arbitrary real orders.

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