Another mathematical optimization models based on assignment problem

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Nové matematické modely pre optimalizáciu pomocou priraďovacieho problému

This contribution is presenting the concrete possibilities of applying the mathematical optimization model based on the graph theory. Method of creating at hematical model for the perfect matching optimized in a special way is described in this paper. Process of optimization is founded on the graph with categorization of edges and special type of the objective function.

Key words: perfect matching, objective function, optimization problem, algorithm, mathematical model.

1. Introduction

Optimization problems derived from solving concrete problems of personnel and human resource management are no longer sufficient to be transformed into classical optimization problems mostly used to solve the problem. We need to find new mathematical models able to cast new looks at the ways of optimizations, i.e. new forms of evaluation and recognition. A frequently used mathematical model applied to discrete structures is the one based on the graph theory and computational complexity.

The combinatorical optimization problem **P** is an ordered pair of (Δ, f), where Δ is the finite set of all the feasible solutions to the given problem and f is an objective function. The theory of computational comlexity differentiates two basic approaches to problem-solving:

- Find such an feasible set of D from family of feasible sets of Δ so that the objective function $f(D)$ is minimal (resp. maximal) on Δ. Such a problem is termed as the optimizational version of the problem.
- Decide whether there exists such an admittable set of D in th the system of feasible Δ sets that the $f(D) \le k$, (resp. $f(D) \ge k$) inequalities holds for the given k. It is known as the recognition version of the problem.

Among the classical problems of combinatorics based on graphs are the ones termed as sum, bottleneck and balanced problems formulated as follows:

Let graph $G = (V, E)$ and the family of feasible sets $\Delta(G)$ be given.. Let each edge $e \in E$ with non-negative weight of $w(e) \in \langle 0, \infty)$ and the classical objective functions are given as:[4]

Sum function: $f_1 = \sum_{e \in D} w(e)$, where $D \in \Delta(G)$,

Bottleneck function: $f_2 = \max_{e \in D} w(e)$, where $D \in \Delta(G)$,

Balanced function: $f_3 = \max_{e \in D} w(e) - \min_{e \in D} w(e)$, where $D \in \Delta(G)$.

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[4] The above oobjective function are the mostly used in mathematical optimization

Then we get the following types of optimization problems:

Sum problem: $f_1(D) \xrightarrow[D\in\mathbf{D}(G)]{} \min$,

Bottleneck problem: $f_2(D) \xrightarrow[D\in\mathbf{D}(G)]{} \min$,

Balanced problem: $f_3(D) \xrightarrow[D\in\mathbf{D}(G)]{} \min$.

2. The problem

These days, the system of rearrangings jobs within a company staff is becoming more and more the central issue. Quite often, the attention is focused on whetrher the staff is receiving equal assignment performing the same amount of work and (if possible), is paid propportionate wages or salaries.

Assume we have available a k number of work groups. Each of them is responsible for an operation within the same 8-hour cycle. This group is then followed by another assigned with the same manufacturing process. The groups are rotating until all of them have participated in the same process. And the same procedure is repeated starting with group one again. Each of the workers are qualified for some of the manufacturing processes the group is involved in. Also, every activity requires certain qualification and a stated workload coefficient, defining the level of load (physical, mental load and similar parameters affecting performance and efficiency) to be born by the employee. Each workgroup is assigned the scope of workload of the group. It is a value stating the difference between the employees with the largest or smallest load.. The smaller the difference the more consolidated is the workload distribution among all the workers in the group.

If the scope is close to zero, then each worker in the group is subjected to almost the same level of workload, corresponding to the just division of labour and more proper renumeration within the group. It is also possible to allow for substitution of the workers whereas the result goes to the credit of the group the workers comes from. The aim is to find such assignment among the workers of several groups and the activities that would ensure that all the jobs get done, while none of the workers would be left without job and all the groups are to function at comparable workloads. In the light of the economic results of the manufacturing process, the money for the payroll will be assigned to groups not to the individuals as, in terms of economy, the group results are easier to measure as the contribution of each individual.

This article is providing solution to the problem of such arrangement of manufacturing processes for the individual workers of the company ensures equality of workload between the groups and the scope of workload is also to be kept balanced. This approach is a justified one, namely, we arrive at qualitatively different-positive results as opposed to those achieved using the classical approach to optimization (the strictly financial point of view). If the criterium of workload maximization per workforce were taken into account only, we could arrive at substantially larger differences within each group. And the absence of the key person within the group would result in steep decline in the quality of the job performed. Using the suggested approach to optimization would bring higher stability and flexibility into human resource management when coping with the unexpected absences of individuals in the process of manufacturing.

3. The mathematical model

Let us remind that combinatorical problem of optimization, **P**, is a pair of (Δ, f), where

$$\Delta = \{D;\ D \textit{ is the feasible solution of the problem } \mathbf{P}\}$$

and f is the objective function. The task is to find the optimal feasible solution D from the family of feasible sets Δ. This article is dealing with the minimization of the objective function f, and, therefore the problem **P** can take the form:

$$f(D) \xrightarrow[D\in\mathbf{D}]{} \min .$$

We are dealing with an optimization task on graphs where the feasible sets are subsets of the edges of the given graph. This problem was formulated in [6]. Let the given graph $G = (V, E)$ and the edge set E is partitioned into disjoint categories $S_1, \dots, S_p$. Each edge $e \in E$ has a defined weight $w(e)$, where $w(e)$ is a non-negative integer and a objective function f taking the form of $L(D)$:

$$L(D) = \max_{1 \le i \le p}\left(\max_{e \in S_i \cap D} w(e) - \min_{e \in S_i \cap D} w(e)\right),$$

where $D \in \boldsymbol{D}(G)$ and suppose that $\max_{e \in \emptyset} w(e) = 0$. [5]

Let us note that the problems with the above defined objective function L are reduced to a classical balanced problem, if the number of categories $p = 1$. If the number of categories is equal to ther number of the sides, then the problem with the objective function L is loosing its importance, as it is reduced to a problem with a constant objective function equal to 0.

The complexity of similar problems were first mentioned by authors such as Averbach, Berman [3], Richey and Punnen [12] and Punnen [11]. Problems with the similiar idea of categorizing edges can be found in works [1, 2, 4, 5, 6, 7, 8, 9, 10].

Let us transform the problems from Chapter 2 into a mathematical model defined in this chapter. The simple graphic illustration of the process in Fig. 1. Let us also assume that the number of workforce available is n_1 and they are divided into a p number of groups ($p \geq 2$). The manufacturing process to be ensured by them is made up of n_2 activities. Each worker is to be assigned what jobs to do at what level of workload. Setting up a graph $G = (V, E)$, wherein the the set of vertices V is given by the unification of sets V_1 and V_2.

The set of V_1 consists of n_1 vertices v_i $i = 1, \dots , n_1$, which correspond to the number of workforce in them., whereas the set V_2 contains $p.n_2$ vertices $v_{k,j}$ $k = 1, \dots , p$, $j = 1, \dots , n_2$, which correspond to the activities to be performed by the workgroups. (i.e. p copies n_2 activities). The $v_{k,j}$ vertex denotes the j-th activity to be performed to the k-th workgroup. The set of sides E is made upe by the of sides connecting the pair of vertices of the garph G in the very moment when one of the vertices is from the set V_1, while the other one is from set V_2 and the i-th worker can perform the j-th activity, while belonging to the p-th workgroup.

Also, the sets of edges E can be added by edges from set V_1 to the sets V_2 and they will link the i-th worker of the k-th workgroup with the j-th activity to be performed by the l-th workgroup. Such a side represents the ability of the i-th worker to act as a substitute in a j-th activity in the l-th workgroup. The sides are assigned weights that correspond to the values of workload coefficients of individual labourers for the given activities. The sides are then divided into p categories so that the sides, of which one end vertex corresponds with the worker from k-th group, were added to the same k-th category S_k for $k = 1, 2, \dots , p$. It gives rise to p categories of sides. If graph G following such a structure-restructuring is bipartial and $|V_1| = |V_2|$, then the structure of the graph G is ready.

Problem **P** is given by graph G, partitioning of the edge set into disjoint categories categories $S_1, \dots , S_p$ and the objective function L. The family of feasible set Δ is made up by perfect matching D of the graph G. If there were no perfect matching in the graph, it would then mean that a worker is missing, a person who is logically unable to carry out the vacant activity. Or, we have to do with someone we are unable to assign any of the activities, as all of them are already being performed. (as it follows from the Dirichlet principle). The case of $|V_1| \neq |V_2|$ will be dealt with at a later time.

The task is to find such a perfect matching of D from the set Δ, for which the value of the objective function L will be minimal in view of the given weights, i.e. in each workgroup in the perfect matching there will be an edge with the largest ($\max_{e \in S_i \cap D} w(e)$) and an edge with the smallest ($\min_{e \in S_i \cap D} w(e)$) coefficient of workload and then the difference is determined. Then we obtain the maximal difference of all the calculated differences i.e. $L = \max_{1 \le i \le p}\left(\max_{e \in S_i \cap D} w(e) - \min_{e \in S_i \cap D} w(e)\right)$.

Optimal solution consists in perfect matching with the minimal value of L, which stands for the lowest possible maximual difference in workload that can be achieved within a workgroup, thereby achieving the balance of workload for all the staff within the group. It is indirectly proving the lelevelness of the distribution of the costs of the workforce for the individuals withing the given group. Based on the results achieved, one can make decisions and adopt new measures in terms of human resource management within the company, the issue no longer dealt with in this article.

[5] For example Δ (G) denotes the set of all the spanning trees, paths between the vertices a and b, of perfect matchings in graph G.

Let us also describe situation when $|V_1| \neq |V_2|$. Suppose $|V_1| < |V_2|$. It means that we lack the workforce to do all the jobs, consequently there is a workgroup which is unable to cover all the activities by labourers. This group is added a fictious vertex representing an imaginary worker. This vertex is then linked, by means of sides, with all the manufacturing activities. All the sides are then assigned the same weight and are given to the 0 category. If upon following this step, it still holds that $|V_1| \neq |V_2|$, then the process is repeated until equality $|V_1| = |V_2|$ is achieved. As a consequence, incase of an optimal, perfect matching D, the imaginary vertices are assigned concrete jobs, where the value of the objective function is known to us. It means that those manufacturing processes are to be ensured by workforce, which will belong to the same category of worker performing activities in the same manufacturing cycle and the value of the workload coeficient will fall between the assigned minimum and maximum assgined value contained in the given matching D. Such an option ensures that the value of the purpose function $L(D)$ will not rise and we start the algorythm to find the new optimum (if it exists).

If there arose the situation when $|V_1| > |V_2|$, then we add the fictionus vertex to the set of V_2 as na imaginary manufaturing process and link it with all the workforce by a side of zero weight. These can also be assigned into the newly-created zero category S_0. On finding the optimum we get the set of the employees assigned the fictious vertex and make up the personnel reserve (*Fig. 1*).

In order to ensure that the given optimization requirements are met, the problem **P** is solved in via an algorythm described in [5] also detailing the complexity of this problem along with the polynomial algorythm of solving the problem **P.**

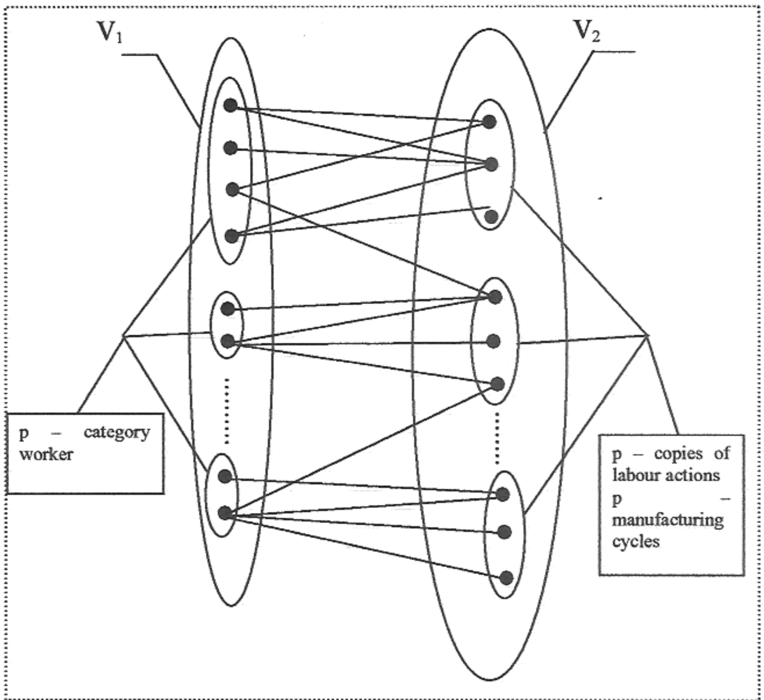

Fig. 1.

4. Conclusion

In this contribution we have illustrated the concrete possibilitiy of applying the mathematical optimization model based on the theory of graphs. Similar approach can be used not only to solve our problem from Chapter 2, but also in other technical or industrial applications. Appart from the mentioned objective function there are available further purpose functions, published by [4, 5, 6, 7, 8, 9], providing optimization models -roblems from various aspects. For example, if we were interested exclusively in the aspect of minimizing costs, it would be then a classical problem of minimal perfect matching with a generally known solution. Application of mathematical models for optimizing similar problems is becoming widely used in practice.

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