Detection and location of nonlinearities in MDOF structural systems

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The study is concerned with the development of a new technique for the detection and location of nonlinearities in MDOF structural systems where multiple nonlinear components may exist due to different faulty conditions. The basis is a concept known as Nonlinear Output Frequency Response Functions and associated properties. The technique determines the location of nonlinear components in MDOF structural systems directly from the system input and output signals, which can be the measurements of sensors fitted into the system for structural health monitoring purpose. Simulation studies demonstrate the effectiveness of the proposed technique.

Key words: *Multi-degree of freedom structural systems, Nonlinearity detection and location, Structural health monitoring*

Introduction

Multi-degree of freedom (MDOF) structural systems can represent a large class of engineering structures in practice including beams, rotor shafts, multi-storey buildings, and bridges etc. Structural health monitoring (SHM) is an important issue associated with these practical engineering structures. MDOF systems can behave nonlinearly simply due to the nonlinear characteristics of one component within the systems, and in many practical cases, such a nonlinear component may represent a fault in the system. Typical examples are beams, as it is well known that beams can behave nonlinearly due to the presence of internal breathing cracks. Therefore, detection and location of nonlinearities in MDOF systems have considerable significance for the health monitoring of a wide range of engineering systems and structures, which can, like beams, behave nonlinearly due to the existence of faults. Many approaches have been proposed to address this issue, which include mutual information and transfer entropy based statistic nonlinearity detection methods, auto-bispectrum analysis, nonlinear system identification technique, and Frequency domain ARX model based technique.

The present study is concerned with the development of a new technique for the detection and location of nonlinearities in MDOF structural systems. The basis is a concept known as Nonlinear Output Frequency Response Functions (NOFRFs), and the properties of NOFRFs for MDOF nonlinear systems. The developed technique can detect the existence and determine the location of nonlinear components in a MDOF structural system from the system input and output signals, which can be the outputs of sensors fitted into the system for SHM purpose. Simulation studies demonstrate the effectiveness of the proposed technique.

MDOF Systems with Nonlinear Components

The systems considered in the present study are described by a typical MDOF structural model as shown in Fig 1 where input force $u(t)$ is applied at mass n. Assume that there are $\bar{L} \geq 0$ nonlinear stiffness and/or damping components in the system, which are component $L(1)$, component $L(2)$,…, and component $L(\bar{L})$, respectively. Denote the restoring forces due to the damping and stiffness of nonlinear component $L(i)$ as $FS_{L(i)}(\Delta)$ and $FD_{L(i)}(\dot{\Delta})$ where $\Delta = x_{L(i)} - x_{L(i)-1}$ and assume that these forces all have a polynomial form, i.e.

$$FS_{L(i)}(\Delta) = \sum_{l=1}^{P} r_{(L(i),l)}\Delta^l \text{ , } FD_{L(i)}(\dot{\Delta}) = \sum_{l=1}^{P} w_{(L(i),l)}\dot{\Delta}^l$$

Then the motion of the considered MDOF structural systems can be described by

$$M\ddot{x} + C\dot{x} + Kx = NF + F(t) \tag{1}$$

where

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$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -c_{n-1} & c_{n-1} + c_n & -c_n \\ 0 & \cdots & 0 & -c_n & c_n \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

$$F(t) = (\overbrace{0,\cdots,0}^{n-1},u(t))' \quad NF = \left(nf(1)\cdots nf(n)\right)'$$

$$nf(l) = \begin{cases} 0 & \text{if } l \neq L(i)-1, L(i),\ 1 \le i \le \bar{L} \\ NonF_{L(i)} & \text{if } l = L(i)-1,\ 1 \le i \le \bar{L} \\ -NonF_{L(i)} & \text{if } l = L(i),\ 1 \le i \le \bar{L} \end{cases} \quad 1 \le l \le n$$

$$NonF_{L(i)} = \sum_{l=2}^{P} r_{(L(i),l)} \Delta^l + \sum_{l=2}^{P} w_{(L(i),l)} \dot{\Delta}^l$$

Equation (1) is the motion governing equation of the system in Fig 1. Obviously, if $\bar{L} \ge 1$, nonlinear components can make the whole system behave nonlinearly. The issue to be addressed in this study is to detect the existence and determine the location of these nonlinear components from the system input output signals. Because, as mentioned in the introduction, the existence of these nonlinear components can indicate the existence of faults in the system such as cracks in beams, detecting and locating these nonlinear components are equivalent to detecting and locating faults and therefore have significant implications in engineering practices.

The NOFRFs of MDOF Systems

The Nonlinear Output Frequency Response Functions (NOFRFs) are a concept recently proposed for the analysis of nonlinear systems in the frequency domain [Lang and Billings 2005]. The concept can be extended to the case of MDOF nonlinear system (1) as:

$$G_{(i,\bar{n})}(\mathrm{j}\omega) = \frac{\displaystyle\int_{\omega_1+,\ldots,+\omega_{\bar{n}}=\omega} H_{(i,\bar{n})}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}}) \prod_{q=1}^{\bar{n}} U(\mathrm{j}\omega_q) d\sigma_{\omega\bar{n}}}{\displaystyle\int_{\omega_1+,\ldots,+\omega_{\bar{n}}=\omega} \prod_{q=1}^{\bar{n}} U(\mathrm{j}\omega_q) d\sigma_{\bar{n}\omega}} \qquad i = 1,\ldots,n;\ \ \bar{n} = 1,\ldots,N \tag{2}$$

where N is the maximum order of system nonlinearity, $U(\mathrm{j}\omega)$ is the spectrum of the system input, and $H_{(i,\bar{n})}(\mathrm{j}\omega_1,\cdots,\mathrm{j}\omega_{\bar{n}})$ represents the $\bar{n}^{\text{th}}$ order generalised frequency response function (GFRF) associated with the displacement of mass i. The nth order GFRF is the extension of the linear frequency response function to the case of nonlinear systems of nth order nonlinearity [George 1959]. By introducing the NOFRF concept, the spectrum of the displacement of mass i of MDOF system (1) can be described as

$$X_i(\mathrm{j}\omega) = \sum_{\bar{n}=1}^{N} G_{(i,\bar{n})}(\mathrm{j}\omega) U_{\bar{n}}(\mathrm{j}\omega) \qquad i=1,\ldots,n \tag{3}$$

where

$$U_{\bar{n}}(\mathrm{j}\omega) = \frac{1/\sqrt{\bar{n}}}{(2\pi)^{\bar{n}-1}} \int_{\omega_1+,\ldots,+\omega_{\bar{n}}=\omega} \prod_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i) d\sigma_{\bar{n}\omega}.$$

Equation (3) is quite similar to the description for the output spectrum of linear systems. This is an important advantage with the introduction of the NOFRF concept.

The properties of the NOFRFs of MDOF systems have been studied in (Peng and Lang, 2007). The results reveal the relationships between the ratios of the NOFFRs of two consecutive masses of different orders. For system (1), these relationships can be summarised as follows.

1. If $1 \le i < L(1)-1$, then

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \tag{4}$$

2. If $L(\overline{L}) \le i < n$, then

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \tag{5}$$

3. If $L(1)-1 \le i < L(\overline{L})$, then

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \ne \cdots \ne \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \tag{6}$$

Equations (3)-(6) provide an important basis for the development of a new technique for the detection and location of nonlinearities in MDOF systems.

The detection and location of nonlinearities in MDOF Systems

The basic idea

From equation (3), the relationship between the output spectra of masses i and i+1 in system (1) can be expressed as

$$X_i(\mathrm{j}\omega) = E_1^{i,i+1}(j\omega)U(\mathrm{j}\omega) + E_2^{i,i+1}(j\omega) + R^{i,i+1}(j\omega)X_{i+1}(\mathrm{j}\omega) \tag{7}$$

where

$$E_1^{i,i+1}(\mathrm{j}\omega) = [G_{(i,1)}(\mathrm{j}\omega) - R^{i,i+1}(j\omega)G_{(i+1,1)}(\mathrm{j}\omega)]$$

$$E_2^{i,i+1}(\mathrm{j}\omega) = \sum_{\tilde{n}=2}^{N}[G_{(i,\tilde{n})}(\mathrm{j}\omega) - R^{i,i+1}(j\omega)G_{(i+1,\tilde{n})}(\mathrm{j}\omega)]U_{\tilde{n}}(j\omega)$$

$$R^{i,i+1}(\mathrm{j}\omega) = \frac{G_{(i,\tilde{n})}(\mathrm{j}\omega)}{G_{(i+1,\tilde{n})}(\mathrm{j}\omega)}\ , \quad 1 < \tilde{n} \le N$$

From properties 1)-3) of the NOFRFs for system (1), it can be derived that if $1 \le i < L(1)-1$ or $L(\overline{L}) \le i < n$, then

$$E_1^{i,i+1}(\mathrm{j}\omega),\ E_2^{i,i+1}(\mathrm{j}\omega),\ and\ R^{i,i+1}(\mathrm{j}\omega)$$

in equation (7) are all independent from the system input and in theory,

$$E_2^{i,i+1}(\mathrm{j}\omega) = 0$$

Otherwise, i.e., when $L(1)-1 \le i < L(\overline{L})$,

$$E_1^{i,i+1}(\mathrm{j}\omega),\ E_2^{i,i+1}(\mathrm{j}\omega),\ and\ R^{i,i+1}(\mathrm{j}\omega)$$

are input dependent and no definite conclusions about the values of these functions can be reached.

These observations imply that the theoretical 0 value of function $E_2^{i,i+1}(\mathrm{j}\omega)$ when $1 \le i < L(1)-1$ and $L(\overline{L}) \le i < n$ is a very important feature of system (1) that can be used to determine $L(1)$ and $L(\overline{L})$ so as to determine the location of the nonlinear components in the system. Based on this idea, a new technique for the detection and location of nonlinearities in MDOF structural system (1) is developed.

The technique

The basic procedure of the technique is described as follows.

a. Excite the system three times using different input forces whose spectra are $U^{(q)}(j\omega)$, q=1,2,3; Measure the responses of all the masses; Then calculate the spectra of the measured responses to obtain

$$X_i^{(q)}(\mathrm{j}\omega),\ i = 1,\ldots,n,\ q=1,2,3.$$

b. Evaluate index IND1 as defined by

$$\mathrm{IND1} = \int_{\omega_1}^{\omega_2} D(\omega)d\omega \tag{8}$$

where $[\omega_1, \omega_2]$ is an interval within the frequency range of the system input, and

$$D(\omega) = \left| \det \begin{bmatrix} U^{(1)}(j\omega) & 1 & X_{i+1}^{(1)}(j\omega) \\ U^{(2)}(j\omega) & 1 & X_{i+1}^{(2)}(j\omega) \\ U^{(3)}(j\omega) & 1 & X_{i+1}^{(3)}(j\omega) \end{bmatrix} \right| \tag{9}$$

Then determine whether or not there are nonlinear components in the system from the value of IND1 as follows: If $\mathrm{IND1} \le \varepsilon$, then there is no nonlinearity or nonlinearity is negligible in the system; Otherwise there is at least one nonlinear component in the system. Here ε is zero in theory but should be a very small number related to the value of IND1 determined in a case where the system basically behaves linearly.

Determine the location of detected system nonlinearities as follows:

(i) Evaluate index IND2 (i,i+1) as defined by

$$\mathrm{IND2}(i,i+1) = \int_{\omega_1}^{\omega_2} \left|\hat{E}_2^{i,i+1}(\omega)\right| d\omega \Big/ \max_{i\in\{1,\dots,n-1\}} \int_{\omega_1}^{\omega_2} \left|\hat{E}_2^{i,i+1}(\omega)\right| d\omega \tag{10}$$

where

$$\hat{E}_2^{i,i+1}(j\omega) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} U^{(1)}(j\omega) & 1 & X_{i+1}^{(1)}(j\omega) \\ U^{(2)}(j\omega) & 1 & X_{i+1}^{(2)}(j\omega) \\ U^{(3)}(j\omega) & 1 & X_{i+1}^{(3)}(j\omega) \end{bmatrix}^{-1} \begin{bmatrix} X_i^{(1)}(j\omega) \\ X_i^{(2)}(j\omega) \\ X_i^{(3)}(j\omega) \end{bmatrix}$$

for i=1,…,n-1.

(ii) Examine IND2 (i,i+1) to find out i_1 and i_2 such that $\mathrm{IND2}(i,i+1) \approx 0$ when $i < i_1$ and $i > i_2$ but $\mathrm{IND2}(i,i+1)\ not \approx 0$ when $i_1 \le i \le i_2$. Then it can be concluded that $L(1) = i_1 + 1$ and $L(\bar{L}) = i_2 + 1$, that is, the system nonlinearities are located between masses i_1 and $i_2 + 1$.

Simulation study

In order to verify the effectiveness of the proposed technique, a simulation study was conducted for the case where a damped 10-DOF system was considered. The linear characteristic parameters of the 10-DOF system are

$$m_1 = \cdots = m_{10} = 1,\ k_1 = \cdots = k_5 = k_{10} = 3.6\times 10^4,$$
$$k_6 = k_7 = k_8 = 0.8\times k_1,\ k_9 = 0.9\times k_1,\ \mu = 0.01,$$

$C = \mu K$. The characteristics of the 4th and 6th spring in the system are nonlinear, that is, $\bar{L} = 2$, $L(1) = 4$ and $L(2) = 6$. The characteristic parameters of the nonlinear springs are

$$r_{(4,1)} = k_1,\ r_{(4,2)} = 8\times k_1^2,\ r_{(4,3)} = 4\times k_1^3,\ r_{(4,l)} = 0, \text{for } l \ge 4$$
$$r_{(6,1)} = k_1,\ r_{(6,2)} = 40\times k_1^2,\ r_{(4,3)} = 16\times k_1^3$$
$$r_{(6,l)} = 0, \text{for } l \ge 4$$

Three sinusoidal force inputs

$$u^{(q)}(t) = \alpha_q \sin(\omega_F t),\ q = 1,2,3$$

where $\omega_F = 2\pi \times 20$, $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 4$ were used in Step a) of the proposed technique to excite the system. In Step b), ε was taken as 10^{-11} which is about the value of IND1 when the system is linear. The evaluated value of IND1 is 2.8611×10^{-7} for the simulated system, clearly indicating that the system is nonlinear. Table 1 shows the values of index IND2(i,i+1) for i=1,…,9 that were obtained in Step c). The results clearly indicate $L(1) = 4$ and $L(2) = 6$, that is, the system nonlinearities are located between masses 3 and 6, which is correct. Therefore the effectiveness of the proposed technique is verified by the simulation study.

Tab. 1. Simulation study result.

IND2 (1,2)	IND2 (2,3)	IND2 (3,4)
0.0006	0.002	**0.6**
IND2 (4,5)	IND2 (5,6)	IND2 (6,7)
0.4	**1.0**	0.03
IND2 (7,8)	IND2 (8,9)	IND2 (9,10)
0.02	0.008	0.01

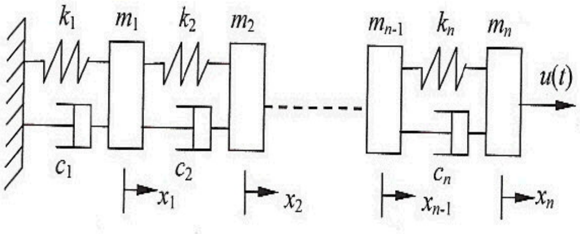

Fig. 1. The considered MDOF structural system.

Conclusions

In this paper, a new technique has been developed for the detection and location of nonlinearities in MDOF structural system (1). The technique is based on the NOFRF concept and associated properties. Its effectiveness has been verified by a simulation study. MDOF structural systems can be used to describe practical engineering structures including beams, rotor shafts, multi-storey buildings, and bridges. Nonlinear components in such systems often represent faults. Therefore, the developed technique has significant potential in fault diagnosis of a wide rage of engineering systems and structures.

References

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