Simple PI/PID Controller Tuning Rules for FOPDT Plants with Guaranteed Closed-Loop Stability Margin

Jan Cvejn [1]

In the paper we present tuning rules for PI and PID controllers and the first order plus dead time (FOPDT) process model. The settings respect both performance and stability indices and provide a high degree of robustness for any value of the dead time parameter. As the performance index we are using the well known modulus optimum criterion, which requires that the amplitude of the closed-loop frequency response is close to one for low frequencies. This criterion produces simple tuning formulas with very favourable properties in time domain. Although optimal values of the parameters are valid for the reference tracking problem, a compensation of the disturbance lag that preserves the stability margin is proposed in the case of the disturbance rejection task.

Keywords: *PID control; process control; time delay*

Introduction

Over decades PID controller has proven to be a very useful instrument in industrial automation. Its application area covers machinery, chemical and food industry, mining industry, automobile and aerospace industry and many others. Due to its simplicity, which allows manual tuning, and capability to succesfully handle even many nonlinear and partially unknown processes it became a standard tool, which is often viewed as universal for achieving practical goals of feedback control.

Dynamics of many industrial processes can be in practice sufficiently modelled by the stable first-order plus dead time (FOPDT) transfer function:

$$S(s) = \frac{K}{Ts+1}e^{-\tau s} \tag{1}$$

where K is the system gain, $T > 0$ is the time constant and $\tau > 0$ is the dead time parameter. The model (1) allows simple experimental identification from the step response, which can be in most cases easily measured. Simple methods based on coincidence in one or more points and more complex methods suitable for noisy data are described in (Åström and Hägglund 1995) and (Kiong et al. 1999).

For tuning PID controllers based on the model (1) many approaches exist, see e.g. (Åström and Hägglund 1995) for description of the most important methods. Early methods were derived from empirical requirements on the step response, such as one-quarter decay ratio (Ziegler and Nichols 1942), (Cohen and Coon 1953), step-response overshoot (Chien et al. 1952) or from integral criterions in time domain with approximation of the dead-time dynamics (Lopez et al. 1967), (Wang et al. 1995). These methods, however, usually work well only for a rather limited range of the ratio τ/T.

Among methods for the model of type (1) without approximation of the delay term the design with given gain and phase margins (Ho et al. 1995) and LQR design (Kiong et al. 1999) should be mentioned. Alternative ways for systems with long time delay include internal model control (Rivera et al. 1986), Smith predictor and λ-tuning (Åström and Hägglund 1995). These approaches, however, require implementation of delay in the control system.

The method for setting up PI controller parameters based on cancellation of the factor $(Ts+1)$ was proposed by Haalman (Haalman 1965). In this method the dead-time dynamics is manipulated without approximation. Good reference tracking performance is achieved, but on the other hand, poor results may be observed for rejection of load disturbances (Åström and Hägglund 1995). In a similar way it is possible to compensate dynamics of the second order plus dead time system by a PID controller.

In this paper we utilize Haalman's idea of pole compensation for designing optimal PI and PID controller parameters for the model (1). The pole compensation fixes the value of one parameter of the controller. The remaining controller parameters can be adjusted to meet analytic design criteria. We show that in this case the modulus optimum criterion leads to a simple choice of the parameters and to a control loop with very favourable properties from practical point of view. Since rather large stability margin is guarranteed for any K, $T > 0$ and $\tau > 0$, the settings provide a high degree of robustness. The results presented here appeared in full context in the paper (Cvejn 2009), where also the settings based on the minimum ISE criterion were analyzed.

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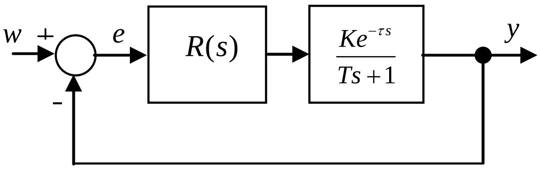

Fig. 1: Control scheme for reference tracking.

The modulus optimum criterion introduced in Oldenbourg and Sartorius (Oldenbourg and Sartorius 1956) requires that the amplitude of the closed-loop frequency response is close to one for low frequencies. If the closed-loop frequency response is decreasing, this condition is analogous to the requirement that the frequencies in the reference input are passed in the broadest possible range. Such a behaviour is desirable especially for the reference tracking cases, because then the closed-loop system is able to respond quickly to changes of the reference input.

Let us write the closed-loop frequency response in the form

$$T(\omega) = \frac{L(i\omega)}{1+L(i\omega)} = \frac{1}{1+1/L(i\omega)} \tag{2}$$

where $L(s)$ is the open-loop transfer function in Laplace transform. If $L(s)$ contains a pole in the origin, which is necessary to achieve asymptotically zero regulation error, for $\omega \to 0$ holds $L(i\omega) \to \infty$ and thus $\lim_{\omega\to 0} |T(\omega)|^2 = 1$. Therefore, it is possible to write $|T(\omega)|^2$ as

$$|T(\omega)|^2 = 1 + H(\omega) \tag{3}$$

where $H(\omega) \in C_\infty$. Maximal flatness of the closed-loop frequency-response modulus is then equivalent to the requirement that

$$n_0\left(H(\omega)\right) \to \max \tag{4}$$

where $n_0\left(H(\omega)\right)$ denotes the index of the first nonzero coefficient in the Taylor expansion of $H(\omega)$.

Besides performance objectives, the design has to respect stability requirements. As the stability margin we consider the distance of the open-loop Nyquist plot from the critical point $[-1,0]$, i.e. the value

$$\gamma = \inf_{\omega\in(0,\infty)} |1+L(i\omega)| \quad \gamma \in [0,\ 1]\,. \tag{5}$$

The reciprocal value of γ is known as the sensitivity function. In general case it is recommended that the sensitivity is in the range from 1.3 to 2 (Åström and Hägglund 1995).

The Controller Design

At first, consider the PI controller case. Consider the reference tracking control problem in Fig. 1. If we compensate the factor $(Ts+1)$ by the PI controller

$$R(s) = K_C\left(1 + \frac{1}{T_I s}\right) \tag{6}$$

where

$$K_C = \frac{\kappa T}{K\tau},\ \ T_I = T \tag{7}$$

the open-loop transfer function is

$$L(s) = \frac{\kappa}{\tau s}e^{-\tau s} \tag{8}$$

where κ is a tuning parameter. The corresponding frequency response can be written as

$$L(\xi) = \frac{\kappa}{i\xi} e^{-i\xi} = \frac{\kappa}{\xi} e^{-i(\xi+\pi/2)} = -\kappa \left(\frac{\sin\xi}{\xi} + i\frac{\cos\xi}{\xi} \right) \tag{9}$$

where $\xi = \tau\omega$ is normalized (dimensionless) frequency. The corresponding Nyquist plot is dependent only on a single parameter κ , which can be adjusted so that sufficient stability margin is guaranteed and performance objectives are fulfilled. In the case of serial PID controller we can put analogously

$$R(s) = K_C \left(1 + \frac{1}{T_I s}\right)(1 + T_D s) = \kappa \frac{T}{\tau K}\left(1 + \frac{1}{Ts}\right)(1 + T_D s) \tag{10}$$

and we easily obtain the corresponding open-loop transfer function

$$L(s) = \kappa \left(\frac{T_D}{\tau} + \frac{1}{\tau s} \right) e^{-\tau s} \tag{11}$$

and the frequency response in the form

$$L(\xi) = \kappa \left(\delta - i\frac{1}{\xi} \right) e^{-i\xi} \tag{12}$$

where $\xi = \tau\omega$ is normalized frequency and $\delta = T_D/\tau$.

Proposition 1. The modulus-optimum settings for the PID controller (10) are:

$$\delta^* = 1/3, \quad \kappa^* = 3/4. \tag{13}$$

and for the PI controller (6):

$$\kappa^* = 1/2. \tag{14}$$

Proof: The closed-loop frequency response (2) square modulus is

$$|T(\xi)|^2 = \frac{1}{1 + \frac{1}{|L|^2}(1 + 2\,\mathrm{Re}\,L)} = \frac{1}{1 + Q(\xi)}. \tag{15}$$

It can be easily verified (Cvejn 2009) that if

$$1 + H(\xi) = \frac{1}{1 + Q(\xi)} \tag{16}$$

where $Q(\xi) \in C_\infty$, it holds

$$n_0\left(H(\xi)\right) = n_0\left(Q(\xi)\right). \tag{17}$$

Since

$$|L(\xi)|^2 = \kappa^2 \left(\delta^2 + \frac{1}{\xi^2} \right) \tag{18}$$

it is

$$n_0\left(Q(\xi)\right) = n_0\left(1 + 2\,\mathrm{Re}\,L(\xi)\right) + 2 = n_0\left(G(\xi)\right) + 1 \tag{19}$$

where

$$G(\xi) = \xi\left(1 + 2\,\mathrm{Re}\,L(\xi)\right). \tag{20}$$

To achieve maximal $n_0\left(Q(\xi)\right)$ and thus maximal $n_0\left(H(\xi)\right)$ we require that the derivatives of $G(\xi)$ are zero for $\xi \to 0$ up to maximal order. After substitution, it is easily found that

$$\lim_{\xi\to 0} \frac{dG}{d\xi} = 1 + 2\kappa(\delta - 1) \tag{21}$$

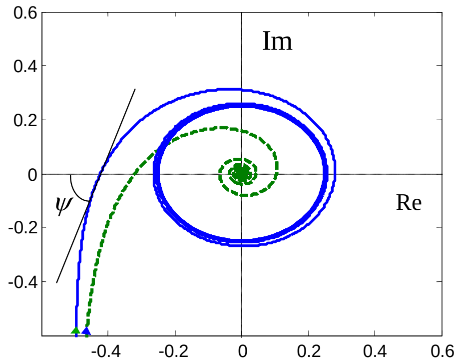

Fig. 2: Open-loop Nyquist plots of proposed settings (solid line - PID controller, dashed line - PI controller).

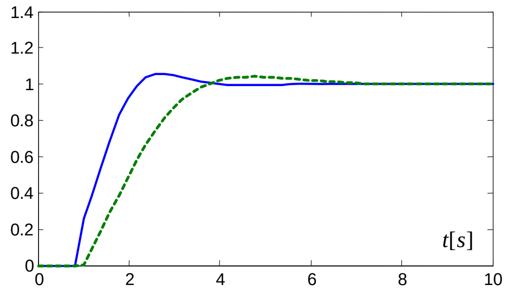

Fig. 3: Step responses (solid line - PID controller, dashed line - PI controller).

$$\lim_{\xi \to 0} \frac{d^3 G}{d\xi^3} = 2\kappa(1 - 3\delta) \tag{22}$$

$$\lim_{\xi \to 0} \frac{d^5 G}{d\xi^5} = 2\kappa(5\delta - 1) \tag{23}$$

$$\lim_{\xi \to 0} \frac{d^{(k)} G}{d\xi^{(k)}} = 0, \;\; k = 0, 2, 4, \ldots \tag{24}$$

Putting (21-22) equal to zero yields $\delta^* = 1/3$ and $\kappa^* = 3/4$. For PI controller, where $\delta = 0$, it follows that the optimal setting is $\kappa^* = 1/2$. $\diamond$

Figures 2 and 3 show the Nyquist plots and corresponding step responses for the proposed settings (the ideal open-loop transfer function (11) with $\tau = 1s$ is considered). Obtained settings obviously have very good quality for most practical purposes - the time response is fast and nearly not oscillating. The overshoot is of about 6 % in the case of PID controller. Figure 4 shows the corresponding dependence of $|T(\xi)|$ on ξ.

The resulting parameters of PI, serial PID and the parallel PID controller

$$R(s) = K_C \left(1 + \frac{1}{T_I s} + T_D s\right) \tag{25}$$

are summarized in Tab. 1.

For the ultimate normalized frequency, where $\arg L(\xi_c) = -\pi$, we easily obtain the equation

$$\xi_c = \pi - \arctan\left(\frac{1}{\delta \xi_c}\right) \tag{26}$$

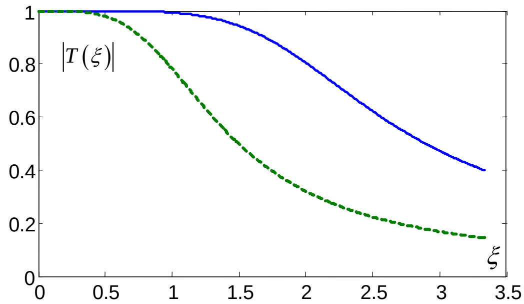

Fig. 4: Dependence $\zeta \rightarrow |T(\xi)|$ (solid line - PID controller, dashed line - PI controller).

Tab. 1: PI / PID controller settings for reference tracking

Controller	K_C	T_I	T_D
PI	$\frac{1}{2}\frac{T}{K\tau}$	T	-
PID (serial)	$\frac{3}{4}\frac{T}{K\tau}$	T	$\frac{1}{3}\tau$
PID (parallel)	$\frac{1}{4K}\left(1+\frac{3T}{\tau}\right)$	$T+\frac{\tau}{3}$	$\frac{\tau}{3+\tau/T}$

which can be solved iteratively. Denote ψ the angle between negative real axis and the Nyquist plot of $L(\xi)$ at the ultimate frequency $\xi = \xi_c$. Geometrical shape of the curve (Fig. 2) enables to construct a lower bound of the stability margin γ using the angle ψ:

$$\tilde{\gamma} = \left(1 - \frac{1}{\alpha}\right)\sin\psi \tag{27}$$

where α is the amplitude margin. $\tilde{\gamma} \approx 0.54$ was obtained for PID controller and $\tilde{\gamma} = 0.57$ for PI controller.

Disturbance Rejection Problem

In most practical cases the reference input is held constant, but the system is excited by external disturbances. We consider that the disturbance influences output through the FOPDT transfer function (see Fig. 5).

It is well known that good tracking performance does not imply efficient disturbance rejection (Åström and Hägglund 1995). The closed-loop transfer function between d and y is

$$S_d(s) = \frac{e^{-\tau_d s}}{T_d s + 1}\,\frac{1}{1 + L(s)} \tag{28}$$

and thus the factor $1/(T_d s + 1)$ will be present in the response regardless of the controller settings, unless it is compensated by a closed-loop zero.

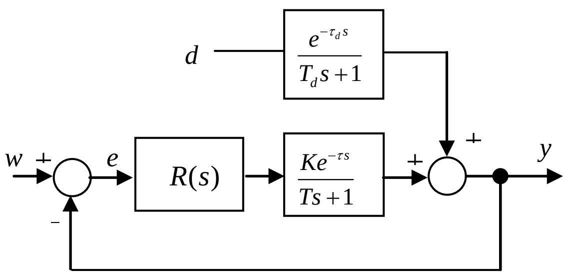

Fig. 5: Control scheme for disturbance rejection.

Since the rise time in the optimal configuration including dead time is not shorter than about 2τ in all the configurations, if $T_d \leq 2\tau$, total dynamics is not affected much adversely by the term $1/(T_d s+1)$ in the input. On the other hand, if $T_d \gg 2\tau$, the factor $1/(T_d s+1)$ can slow down the response significantly.

The term on the right in (28) corresponds to the transfer function of the regulation error at the reference tracking problem and thus the optimal disturbance rejection problem is analogous to the problem of optimal tracking the reference signal with L-transform

$$W(s) = \frac{e^{-\tau_d s}}{T_d s+1}\frac{1}{s}. \tag{29}$$

Therefore, one way how to achieve good performance is to sufficiently decrease T_d, while keeping the other parts of the closed-loop transfer function unchanged. If d is not measured, both these objectives probably cannot be fulfilled. Below a compensation that reduces T_d to T_d' and simultaneously approximately preserves the stability margin is proposed. It is possible to assume that in this case the performance will not be much degraded.

If we choose the controller in the form

$$R_d(s) = \frac{T_d}{T_d'}\frac{T_d' s+1}{T_d s+1}R(s) \tag{30}$$

where $R(s)$ is the controller tuned for the reference tracking and $T_d' < T_d$, the closed-loop transfer function is

$$S_d(s) = \frac{e^{-\tau_d s}}{T_d s+1}\frac{1}{1+\frac{T_d}{T_d'}\frac{T_d' s+1}{T_d s+1}L(s)} = \frac{T_d'}{T_d}\frac{e^{-\tau_d s}}{T_d' s+1}\frac{1}{\frac{T_d'}{T_d}\frac{T_d s+1}{T_d' s+1}+L(s)}. \tag{31}$$

However, such a reduction of T_d at the same time decreases the stability margin. Denote $L_d(\xi)$ the open-loop frequency response if the controller (30) is used. If we assume that $\tau/\left(T_d'\xi\right) \ll 1$ and $T_d \geq T_d'$, holds

$$L_d(\xi) = \frac{T_d}{T_d'}\frac{iT_d'\xi/\tau+1}{iT_d\xi/\tau+1}L(\xi) \approx \left(1-i\frac{1}{\xi r_d}\right)L(\xi) \tag{32}$$

where

$$r_d = \frac{1}{\tau}\left(\frac{1}{T_d'}-\frac{1}{T_d}\right)^{-1}. \tag{33}$$

We determine the parameter r_d from the condition

$$\frac{\tilde{\gamma}-\tilde{\gamma}_d}{\tilde{\gamma}} = h \tag{34}$$

where $\tilde{\gamma}$, $\tilde{\gamma}_d$ are the lower estimates of the stability margin given by formula (27) for the original controller and the modified controller (30), respectively, and h is a sufficiently small chosen constant. Equation (34) is solved iteratively, see (Cvejn 2009) for complete explanation. T_d' is then obtained from

$$T_d' = \left(\frac{1}{r_d\tau}+\frac{1}{T_d}\right)^{-1}. \tag{35}$$

The value h should be chosen so that the stability margin be approximately preserved, but since small h leads to large T_d', a compromise has to be looked for. A good choice seems to lie in the range $h \in [0.05,\ 0.08]$. The results corresponding to $h = 1/16$ are $r_d = 5.92$ for PI controller and $r_d = 3.91$ for PID controller. Note that for $T_d/\tau \to 0$ we obtain $R_d(s) \to R(s)$ and the controller is the same as for the reference tracking problem.

Figure 6 shows the open-loop Nyquist plots after the compensation for PID controller, for $T_d \to \infty$, $T_d = 5\tau$, $T_d = \tau$ and $T_d = 0$ (i.e. without the compensation). A disadvantage of the proposed compensation is that additional term $(T_d' s+1)/(T_d s+1)$ is needed. However, if $T_d = T$ (this is also the case when the disturbance influences the system input), the factor $1/(Ts+1)$ of the system need not be directly compensated. Instead, the PID controller takes the form

$$R_d(s) = \frac{T}{T_d'}\frac{T_d' s+1}{Ts+1}\frac{\kappa T}{\tau K}\left(1+\frac{1}{Ts}\right)(T_D s+1) = \frac{\kappa T}{\tau K}\left(1+\frac{1}{T_d' s}\right)(T_D s+1) \tag{36}$$

which is similar to (10). Note that the integral term of the controller, which is needed to achieve zero regulation error, here plays the additional role of compensating the disturbance lag.

Tab. 2: PI / PID controller settings for disturbance rejection

Controller	K_C	T_I	T_D
PI	$\frac{1}{2}\frac{T}{K\tau}$	$\left(\frac{1}{5.9\tau}+\frac{1}{T}\right)^{-1}$	-
PID (serial)	$\frac{3}{4}\frac{T}{K\tau}$	$\left(\frac{1}{3.9\tau}+\frac{1}{T}\right)^{-1}$	$\frac{1}{3}\tau$
PID (parallel)	$\frac{1}{4K}\left(1+3.26\frac{T}{\tau}\right)$	$\left(\frac{1}{3.9\tau}+\frac{1}{T}\right)^{-1}+\frac{1}{3}\tau$	$\left(\frac{3.26}{\tau}+\frac{1}{T}\right)^{-1}$

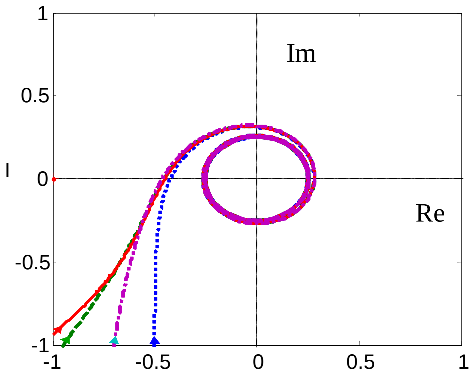

Fig. 6: Open-loop Nyquist plots after compensation, PID controller for $T_d \to \infty$ (solid line), $T_d = 5\tau$ (dashed), $T_d = \tau$ (dash-dotted) and $T_d = 0$ (dotted).

The corresponding settings of PI and PID controllers for input disturbance rejection, $h = 1/16$, are summarized in Tab. 2.

Experimental Comparisons

At first, we consider the reference tracking problem, where the reference signal is not known in forward. Although many of PID tuning formulas for the model (1) are available, most of them are applicable only for a rather limited range of the ratio $\theta = \tau/T$. Usually, it is required that $\theta \geq 0.1$ and $\theta \leq 1$ or $\theta \leq 2$. The minimum ISE, IAE and ITAE tuning rules by Wang et al. (Wang et al. 1995), where the recommended range of θ is $\theta \in [0.05,\ 6]$, are among the exceptions. Note that the modulus-optimum (MO) settings we propose admit any positive value of θ.

Figures 7 and 8 show the reference signal step responses for the plants

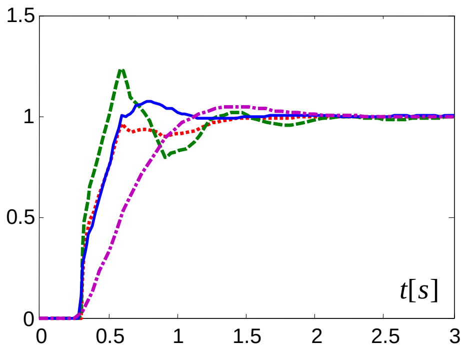

Fig. 7: $F_1(s)$ reference tracking, step response. Settings: MO-PID (solid line), MO-PI (dash-dotted), Chien (dashed), Wang (dotted).

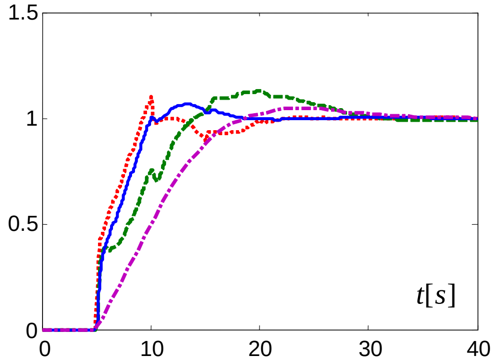

Fig. 8: $F_2(s)$ reference tracking, step response. Settings: MO-PID (solid line), MO-PI (dash-dotted), Chien (dashed), Wang (dotted).

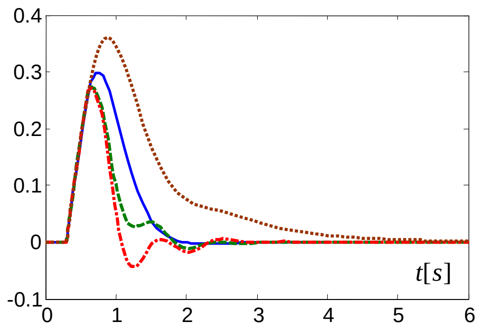

Fig. 9: $F_1(s)$ load disturbance, step response. Settings: MO (solid line), Chien (dashed), Lopez (dash-dotted), Haalman (PI) (dotted).

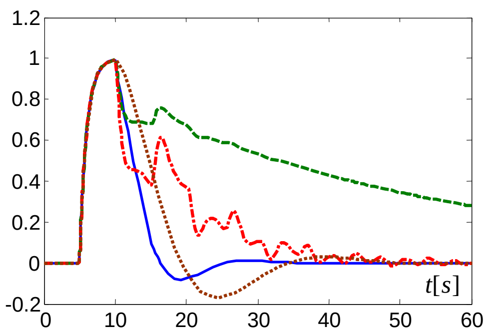

Fig. 10: $F_2(s)$ load disturbance, step response. Settings: MO (solid line), Chien (dashed), Lopez (dash-dotted), Haalman (PI) (dotted).

$$F_1(s) = \frac{1}{s+1}e^{-0.3s}, \;\; F_2(s) = \frac{1}{s+1}e^{-5s} \tag{37}$$

and parallel (ideal) PID controller tuned by using Wang IAE formulas (Wang et al. 1995), well known formulas by Chien et al. (Chien et al. 1952) for 20% step-response overshoot and MO settings of PI and PID controller. Chien settings for large θ result in too slow response, which could be expected, since recommended range of θ is $\theta \in (0.11, \; 1)$. We also tested well known Ziegler-Nichols (Ziegler and Nichols 1942) formula, which for $F_1(s)$ give a rather oscillating response with about 75% overshoot and for $F_2(s)$ slow and overdamped response.

For the disturbance rejection problem we consider that the disturbance influences the system input, i.e. $T_d = T$. Figures 9 and 10 show the load disturbance step responses for the plants $F_1(s)$ and $F_2(s)$ and parallel PID controller tuned according to the minimum IAE formulas by Lopez et al. (Lopez et al. 1967), disturbance rejection formulas with 20% overshoot by Chien et al. (Chien et al. 1952), Haalman PI controller settings and MO-tuned PID controller with the input disturbance compensation. Obviously, Lopez and Chien formulas, recommended for $\theta \in [0.1, \; 1]$, are not suitable for large θ. Ziegler-Nichols settings give responses very similar to Chien settings, in both the cases. In all the cases the proposed settings give very satisfactory results.

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