Calculation of a Shock Response Spectra

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As it is stated in the ISO 18431-4 Standard, a Shock Response Spectrum is defined as the response to a given acceleration acting at a set of mass-damper-spring oscillators, which are adjusted to the different resonance frequencies while their resonance gains (Q-factor) are equal to the same value. The maximum of the absolute value of the calculated responses as a function of the resonance frequencies compose the shock response spectrum (SRS). The paper will deal with employing Signal Analyzer, the software for signal processing, for calculation of the SRS. The theory is illustrated by examples.

Keywords: *Shock response spectrum, shock, single degree of freedom system, measurements, ramp invariant method.*

Introduction

Shock Response Spectrum (SRS) analysis was developed as a standard data processing method in the early 1960's. SRS was used by the U.S. Department of Defense. Now this signal processing method is standardized by ISO 18431-4 Mechanical vibration and shock –Signal processing –Part 4: Shock response spectrum analysis [1].

Let it be assumed that many small instruments or other parts as substructures (bodies) are mounted to the base structure (baseplate) of a mechanical system [2]. The mounting elements are flexible and could be described by stiffness and damping parameters. The mass of the mentioned substructure creates the mechanical oscillator, which is a single-degree-of-freedom (SDOF) system. Now, we suppose that a vibration transient is excited on the base structure as a consequence of either its environment or normal operation. Many sources of excitation could be described: Drop impact during handling, explosive bolts on aerospace structures, bolted joints suddenly opening and closing with an impact, reciprocating engine fuel explosions inside cylinders, etc. For example we could assume that vibration transient results from the dropping of a mechanical system onto a concrete surface while producing an impact load on one corner of the base structure. The impulse force acting to substructures excites lightly damped oscillations. The peak value of the substructure acceleration may cause destruction due to the large inertia force. It is required that the mentioned substructures, in fact equipments and instruments, have to survive mechanical shocks. The impulse force and shock exposure have to be analyzed.

Two main parameters of the SDOF system, which are specifying free oscillation, are the substructure natural frequency and damping. There are many measures of describing damping. This paper prefers the damping ratio $Q = 1/(2\xi)$ and the quality factor Q, which is approximately equal to a resonance gain of the lightly damped systems. For a normalized value of the quality factor Q the substructure natural frequency f_n plays the key role. In this case the substructure natural frequency determines a peak acceleration, which is excited by an arbitrary transient acceleration input, such as shock or impulse force load. The dependence of the acceleration peak value (maximum or minimum) on the substructure natural frequency in the form of a diagram is called a shock response spectrum (SRS) of the impulse force. SRS of the impulse force or acceleration forecasts the excited substructure acceleration in dependence on the substructure resonance frequency. As it was stated above, SRS assumes that the impulse force is applied as a common base input (base plate) to a group of independent hypothetical SDOF systems (bodies), which are only different in the natural frequency, see Fig. 1. Damping ratio is typically fixed at a constant value, such as 5%, which is equivalent to the quality factor Q, which equals to 10. An example of the vibration amplitude of the SDOF system as a response to the input impulse force is demonstrated in Fig. 1. The peak values of the amplitude determine SRS of the associated impulse force. In a general sense SRS allows to compare the dynamic effects of various shock sources through their effect on the set of the SDOF systems.

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(Review and revised version 15. 09. 2011)

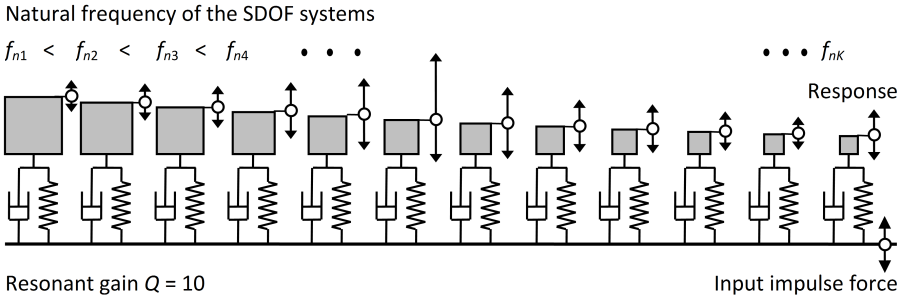

Fig. 1: Group of single-degree-of-freedom systems .

Single-Degree-of-Freedom Model

The equation of motion, which is describing oscillation of the single-degree-of-freedom (SDOF) system, shown in Fig. 2, is written in the form

$$m\ddot{x}_1 = -c(\dot{x}_2 - \dot{x}_1) - k(x_2 - x_1) \tag{1}$$

where x_1 is a base structure (baseplate) displacement and x_2 is a response displacement of a single-degree-of-freedom system (body), m, c, k are mass, damping parameter and stiffness, respectively. The corresponding velocity as a time function is designated by v_1 and v_2 and acceleration is designated by a_1 and a_2. The Laplace transfer function relating

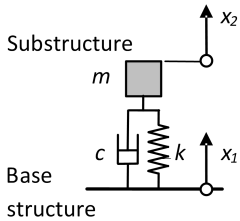

Fig. 2: SDOF, a body (substructure) flexibly mounted on a baseplate (base structure).

the substructure mass displacement to the base displacement is as follows

$$G(s) = \frac{X_2(s)}{X_1(s)} = \frac{cs+k}{ms^2+cs+k} \tag{2}$$

Assuming the zero initial condition, the relationship between the Laplace transform of displacement and velocity and relationship between the same transform of velocity and acceleration are as follows

$$V_1(s) = sX_1(s), \quad V_2(s) = sX_2(s), \quad A_1(s) = sV_1(s), \quad A_2(s) = sV_2(s) \tag{3}$$

where

$$L\{x_1(t)\} = X_1(s),\ L\{v_1(s)\} = V_1(s),\ A\{a_1(t)\} = A_1(s),\ L\{x_2(t)\} = X_2(s),\ L\{v_2(s)\} = V_2(s),\ A\{a_2(t)\} = A_2(s) \tag{4}$$

Alternatively the Laplace transfer function relating the substructure mass velocity to the base velocity and the substructure mass acceleration to the base acceleration are as follows

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{A_2(s)}{A_1(s)} = \frac{cs+k}{ms^2+cs+k} \tag{5}$$

All the transfer functions are of the same form. Introducing the SDOF system natural frequency f_n, Q factor (resonance gain) or damping ratio ξ (fraction of critical damping), we obtain

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \quad Q = \frac{\sqrt{km}}{c}, \quad \xi = \frac{1}{2Q} = \frac{c}{2\sqrt{km}} \tag{6}$$

The Laplace transfer function relating the base acceleration to the substructure mass acceleration could be written in the form

$$\frac{A_2}{A_1} = \frac{\frac{\omega_n s}{Q} + \omega_n^2}{s^2 + \frac{\omega_n s}{Q} + \omega_n^2} = \frac{R_1}{s - s_1} + \frac{R_1^*}{s - s_1^*}, \tag{7}$$

where R_1 is a residue, s_1 is a pole and R_2, s_2 are corresponding complex conjugate quantities.

For a given quality factor Q, natural angular frequency f_n and base structure acceleration a_1 in the form of the time function, it is theoretically possible to calculate the acceleration response a_2 in the form of the time function as well and consequently to determine the peak value (either maximum or minimum)) of this time function. The problem is that the input acceleration is in the form of the sampled signal with the sampling interval $T_S = 1/f_S$, not in the form of a continuous signal. The transfer function (7) has to be approximated by the Z-transform function, i.e. to be transformed into a discrete system (digital filter).

Discrete Approximation of a Continuous Transfer Function

This part of the paper deals with a mapping of the s-plane (continuous time) to the z-plane (discrete time) saving the important properties of the continuous system after transformation into a discrete system of the same order. In the case of the transfer functions (2) and (5), which are identical, it concerns the resonance frequency and resonance gain. The approximation of the differential equation by the difference equation can be performed by using the numerical integration of the sampled function using trapezoidal rule. This method results in the bilinear transform, which distorts the frequency due to the fact that the infinity frequency interval of the continuous system is mapped onto the finite frequency interval of the discrete system. To avoid the frequency distortion the bilinear transform is extended by a frequency correction saving the identity of the transfer function for the resonance frequency

$$G^*(z) = G(s)\,\Big|_{s=\frac{2\pi f_P}{\tan(\pi \frac{f_P}{f_S})}\frac{z-1}{z+1}} \tag{8}$$

where f_S is the sampling frequency and f_P is the prewarping frequency, both in Hz. The frequency responses, before and after mapping, match exactly for the prewarping frequency. It is therefore desirable to substitute the prewarping frequency by the resonance frequency.

The approximation, which is preferred by the ISO 18431-4 Standard, is called the Ramp Invariant Method and was designed by Ahlin, Magnevall, and Josefsson [3]. This method is based on the approximation of the s-plane transfer function

$$G_1(z) = \frac{1}{s+a} \tag{9}$$

of the same form as (7) (except a residue) by the z-plane transfer function

$$G_1(z) = \frac{\beta_{10} + \beta_{11} z^{-1}}{1 + \alpha_{11} z^{-1}} \tag{10}$$

The parameter a determines the pole $(-a)$ of transfer function (9) and parameters β_{10}, β_{11}, α_{11} are coefficients of the approximating difference equation of the first order. The impulse response, corresponding to the linear system (9), is as follows

$$g_1(t) = L^{-1}\{G_1(s)\} = \exp(-at) \tag{11}$$

In order to derive the general formula for a response of the system with the transfer function (9) the general input and output signal are introduced. Let $y_2(t)$ be an output signal of a continuous linear system with input $y_1(t)$ and the impulse response $g_1(t)$ can be computed by using the convolution integral

$$y_2(t) = \int_0^t g_1(t-\tau) y_1(\tau) d\tau \tag{12}$$

The measurement of continuous signals results in a sequence of samples in the discrete time $t = nT_S$, where $n = 0, 1, 2, \ldots$ For the linear system, which is described by the impulse response (11), the value of the output signal at the time $t = (n+1)T_S$ could be evaluated by using the mentioned convolution integral

$$\begin{aligned} y_2((n+1)T_S) &= \int_0^{(n+1)T_S} \exp\left(-a((n+1)T_S - \tau\right) y_1(\tau) d\tau = \\ &= \exp\left(-aT_S\right) \left(\int_0^{nT_S} \exp\left(-a(nT_S - \tau\right) y_1(\tau) d\tau + \int_{nT_S}^{(n+1)T_S} (\ldots) d\tau \right) = \\ &= \exp\left(-aT_S\right) y_2(nT_S) + \exp\left(-aT_S\right) \int_0^{T_S} \exp\left(au\right) y_1(u + nT_S) du \end{aligned} \tag{13}$$

The continuous function $y_1(t)$ in between two adjacent samples may be approximated either by a constant value or by a ramp or by any suited function. Undependably of the approximation of the unknown values of the function in between samples $y_1(nT_S)$ and $y_1(nT_S+T_S)$, the value of $y_2(nT_S+T_S)$ is computed by using a formula

$$y_2((n+1)T_S) = \exp(-aT_S)y_2(nT_S) + \beta_{10}y_1((n+1)T_S) + \beta_{11}y_1(nT_S) \tag{14}$$

which corresponds to the discrete transfer function

$$G_1^*(z) = \frac{\beta_{10}+\beta_{11}z^{-1}}{1-\exp(-aT_S)z^{-1}} \tag{15}$$

As it was mentioned before, the approximation by the ramp is preferred [3]

$$y_1(t) = y_1(nT_S) + \frac{y_1((n+1)T_S) - y_1(nT_S)}{T_S}(t-nT_S), \qquad nT_S \le t < (n+1)T_S \tag{16}$$

This approximation enables to compute the value of the parameters β_{10} and β_{11}.

Now let us return to the kinematic quantities as displacement, velocity and acceleration, which were designated by $x_1(t)$, $v_1(t)$ and $a_1(t)$ respectively, for the base structure and by $x_2(t)$, $v_2(t)$ and $a_2(t)$ respectively for the substructure. The formula (7) demonstrates the decomposition of the second order system into two parallel systems of the first order. The continuous transfer function (9) can be approximated by the discrete transfer function (15). Therefore, the approximation transfer function of the second order system can be composed of two discrete transfer functions (15), but each of them is multiplied by the corresponding residue. The digital filter corresponding to SDOF system responses is the second order filter of the IIR (Infinite Impulse Response) type, with the general Z-transform expression

$$G^*(z) = \frac{A_2(z)}{A_1(z)} = \frac{\beta_{20}+\beta_{21}z^{-1}+\beta_{22}z^{-2}}{1+\alpha_{21}z^{-1}+\alpha_{22}z^{-2}} \tag{17}$$

where β_{20}, β_{21}, β_{22}, α_{21}, α_{22} are parameters of the second order filter.

The transfer function (17) corresponds to the second order difference equation relating the samples $x_1(nT_S)$ and $x_2(nT_S)$ of the base structure and substructure displacements, respectively

$$x_2(nT_S) = \beta_{20}x_1(nT_S) + \beta_{21}x_1((n-1)T_S) + \beta_{22}x_1((n-2)T_S) - \alpha_{21}x_2((n-1)T_S) - \alpha_{22}x_2((n-2)T_S) \tag{18}$$

where n is an integer number of the same meaning as in formula (13).

The displacement samples can be substitute by the velocity, $v_1(nT_S)$ and $v_2(nT_S)$, or acceleration, $a_1(nT_S)$ and $a_2(nT_S)$, samples.

This chapter describes how to approximate the continuous transfer function by the discrete transfer function. Detailed description of a procedure how to create a formula for calculation of the filter parameters is omitted. The value of the digital filter parameters is calculated by formulas given below, which are tabulated in the ISO 18431-4 Standard

$$\beta_{20} = 1 - \exp(-A)\frac{\sin B}{B} \tag{19}$$

$$\beta_{21} = 2\exp(-A)\left[\frac{\sin B}{B} - \cos B\right] \tag{20}$$

$$\beta_{22} = \exp(-2A) - \exp(-A)\frac{\sin B}{B} \tag{21}$$

$$\alpha_{21} = -2\exp(-A)\cos B \tag{22}$$

$$\alpha_{22} = \exp(-2A) \tag{23}$$

where

$$A = \frac{\omega_n T_S}{2Q}, \quad B = \omega_n T_S\sqrt{1-\frac{1}{4Q^2}} \tag{24}$$

Comparison of Frequency Responses Calculated by Different Methods

For given parameters such as Q and f_n or $\omega_n = 2\pi f_n$, one can calculate the frequency response which is corresponding to the transfer function of the system (7) and approximation of this continuous system by a discrete linear system using the ramp invariant method and the bilinear transformation taking into acount the frequency distortion by using the formula (8). Frequency response of the continuous system is calculated in the frequency range to the Nyquist frequency $(f_S/2)$ by substituting $s = j\omega$. The substitution $z = \exp(j\omega T_S)$ is used for calculation of the frequency response of the discrete system. According to Fig. 3, no difference between the resonant frequency of the continuous and discrete systems is observed. The same result can not be obtained for a natural frequency f_n which is approaching the Nyquist frequency. There is the only difference between the frequency responses for frequencies near the Nyquist frequency and higher than the resonant frequency which does not affect the calculation of the SRS for which is important matching the frequency responses for the natural frequency.

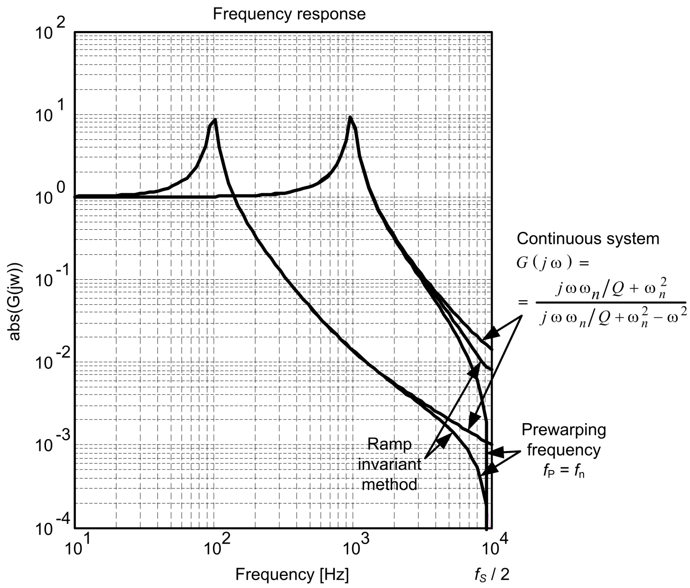

Fig. 3: Comparison of frequency responses.

Transfer Function for Relative Velocity and Displacement Response

For some substructure, the difference of the velocity or displacement between the base structure and the mentioned substructure, which is excited by the base structure impulse load, is more dangerous than the substructure acceleration. The other transfer functions, which are relating the velocity or displacement difference to the base structure acceleration, are shown in table 1. The corresponding filter coefficients could be found in [1].

Tab. 1: Mechanical impulse

Relative velocity response spectrum	Relative displacement response spectrum
$\frac{V_2-V_1}{A_1} = \frac{-ms}{ms^2+cs+k}$	$\frac{X_2-X_1}{A_1} = \frac{-m}{ms^2+cs+k}$
$\frac{X_2-X_1}{A_1}\omega_n = \frac{-m\omega_n}{ms^2+cs+k}$	$\frac{X_2-X_1}{A_1}\omega_n^2 = \frac{-m\omega_n^2}{ms^2+cs+k}$

Software Tools for SRS Calculation

There are many software tools for signal analysis, for example Matlab, which is a high-performance language for technical computing. The disadvantage is that the high frequency sampling is not integrated in the unified environment and a user has to import the input data. On the other hand there is a software tool, which is specialized only on signal measurements and recordings without signal processing. Signal Analyzer, the VSB - Technical University of Ostrava indoor software, extended by scripts is an experiment how to overcome the mentioned disadvantage. Signal Analyzer is software intended to support laboratory measurements. The recorded data may be immediately tested by an analyzer virtual instrument. The script language is described in paper [4].

```
'Shock Response Spectrum';
'CrLf';
ymax=[];
ff=[];
fmin=1;
fmax=1000;
n=90;
qv=(fmax/fmin)^(1/n);
T=1/get(input1,'freq');
Q=10;
for(i=0;i<n;i=i+1)
{
  fn=fmin*qv^i;
  ff=[ff,fn];
  wn=2*pi*fn;
  A=wn*T/2/Q;
  B=wn*T*sqr(1-1/4/Q/Q);
  b0=1-exp(-A)*sin(B)/B;
  b1=2*exp(-A)*(sin(B)/B-cos(B));
  b2=exp((-2)*A)-exp(-A)*sin(B)/B;
  a1=(-2)*exp((-1)*A)*cos(B);
  a2=exp((-2)*A);
  BB=[b0,b1,b2];
  AA=[-a1,-a2];
  y=filter(input1,BB,AA);
  yy=max(y);
  ymax=[ymax,yy];
};
save(ff);
save(ymax);
```

Fig. 4: Signal Analyzer script for calculation of SRS.

Half Sine as an Input Function

The popular testing function for evaluation of SRS is a half-sine function as the time history of the input acceleration. The mentioned testing signal and the result of calculation of the SDOF response in the form of SRS are shown in Fig. 5.

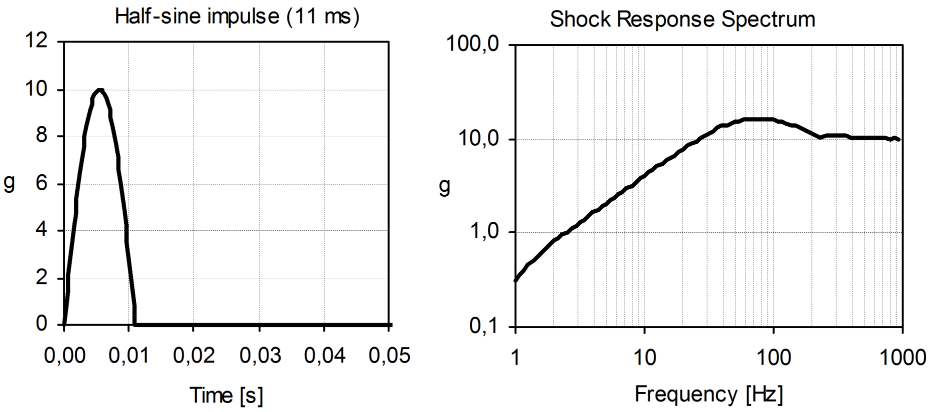

Fig. 5: Shock Response Spectrum of a half-sine impulse (interval of 11 ms).

Acceleration response is represented on the y-axis, and natural frequency of any given hypothetical SDOF is shown on the x-axis. The y-axis of both the plots is in multiple of the free fall acceleration. Any units of acceleration are admissible as well. SRS covers frequency over three decades. 30 responses of the SDOF system in each decade are calculated for different values of the natural frequency. The natural frequencies form a geometric sequence with common ratio 1.080. Therefore SRS in Fig. 5 seems to be a continuous function. The maximum value of the acceleration response is reached for the substructure natural frequency approximately around of 70 Hz. For a given impulse it is dangerous to tune the resonance frequency of the substructure, which is attached to the base structure, to the value in the interval ranging from 30 to 200 Hz due to the large impact loading if the resonance frequency greater than 200 Hz causes the impact loading to be at the margin of acceptability.

Measurement Example

An examples of the shock measurements and calculation SRS is added to demonstrate the described calculation of SRS. The example shows effect of hammer tip material on the shock response spectrum. The hammer tip was made from various materials, such rubber, plastic and aluminum. The base structure is a steel cylinder of the 1.55 kg mass, which is hanging on flexible tapes allowing displacement in axial direction of the cylinder. It is assumed that some substructure is mounted on the basic structure. An accelerometer was attached to a cylindrical body, whose acceleration is measured. Because that the hammer is intended to excite a mechanical structure for experimental modal analysis it is equipped with a force transducer. The hammer hit to the steel cylinder excites both the force and base structure acceleration.

The time history of the impulse force is shown in Fig. 6 while the corresponding acceleration responses are shown in Fig. 7. It must be emphasis that the vibration responses are needed as the input data for computing of the shock response spectrum in order to assess mechanical loading the substructures, which are mounted on the base structure, when a shock occurs. The mechanical impulse in table 2, as an integral of force with respect to time, serves to the comparison of the mentioned test hit intensity.

Tab. 2: Mechanical impulse

Material	rubber	plastic	aluminum
Impulse [Ns]	0.1506	0.1311	0.0661

The shock response spectrum was computed for unified values of impulse with respect to the reference impulse of the hammer hit with the rubber tip. According to the impulse-momentum theorem, the resulting momentum of the testing mass, i.e. velocity of the testing mass, is identical for all the hammer tip materials. The result of the SRS calculation is shown in Fig. 8. As it is evident the maximum excitation for the rubber tip of the hammer is reached at the resonance frequency of 250 Hz, while the plastic tip excites the maximum vibration response at 2000 Hz and the aluminum tip is able to excite the maximum response of the mass vibration at 3500 Hz. The peak at approximately 2 Hz corresponds to the natural frequency of the mass-spring system. SRS informs on the effect of the impulse force on the vibration frequency and can be used to compare individual hits by the hammer for the experimental modal analysis.

The effect of the impact force on the single-degree-of-freedom system, which is tuned to a resonance frequency, describes the properties of the mentioned force better than its time history of the impact force or acceleration. The reason for this statement is that SRS draws attention to particularly sensitive frequency at which can be excited extremely high amplitude of oscillation

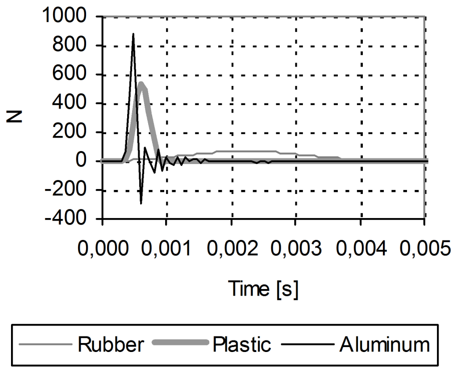

Fig. 6: Impulse force acting on the base structure.

Conclusion

The paper presents the method of calculation of a shock response spectrum, which is corresponding to an acceleration exciting the resonance vibration of substructures. SRS determines the maximum or minimum of the substructure acceleration response as a function of the natural frequencies of a set of the single-degree-of-freedom systems modeling the mentioned substructures.

The vibration (or shock) is recorded by signal analyzer in the digital form, commonly as acceleration. The transfer function of the single-degree-of-freedom system is approximated by an IIR digital filter in such a way that the filter response to the sampled acceleration may be easily calculated. This shock response spectrum shows how the individual components of the impulse signal excite the mechanical structure to vibrate.

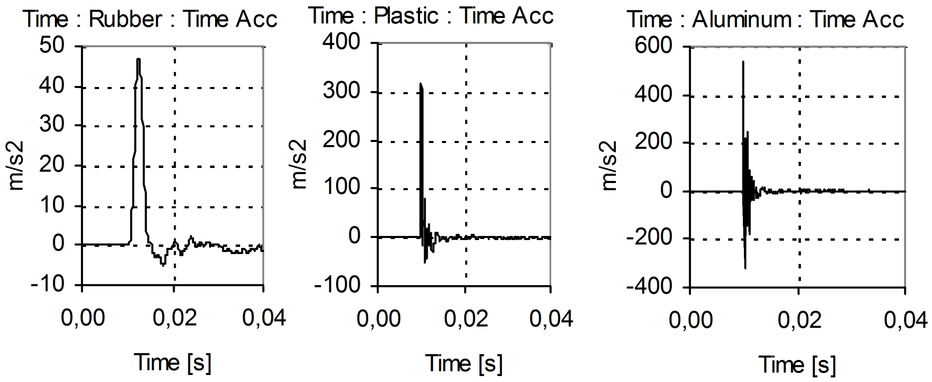

Fig. 7: Impulse force acting on the base structure.

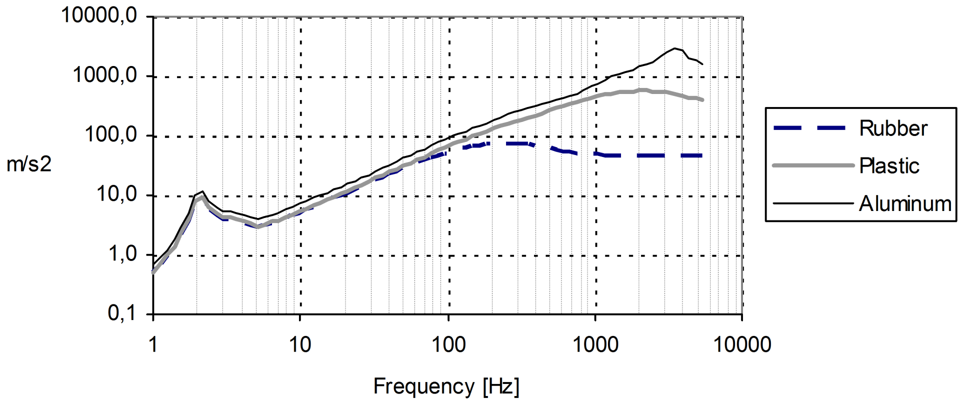

Fig. 8: SRS of the shock, which is excited by the hammer.

Acknowledgement
The work was supported by research project GACR No 102/09/0894.

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