Unified Approach to Analog and Digital Two-Degree-of-Freedom PI Controller Tuning for Integrating Plants with Time Delay

***Miluse Viteckova*[1], *Antonin Vitecek*[2], *Marek Babiuch*[3]**

The article describes the unified approach to the tuning of the analog and digital PI controller with two-degree-of-freedom by the multiple dominant pole method for integrating plants with a time delay on the basis of D-transform. The approach is fully analytical and it enables so tuning that the servo and regulatory responses are non-oscillatory without overshoots. The uses are shown in the example.

Keywords: *2DOF controller, PI controller, integrating plant, time delay*

Introduction

The tuning of the PI controller for integrating plants with a time delay belongs to demanding problems of the control theory and practice. It is caused by the type of the control system $q = 2$. Predisposition to instability and presence of big overshoots are main problems [1, 2, 3, 4].

The integrating plus time delay (IPTD) plant with the L-transfer function.

$$G_P(s) = \frac{k_1}{s}\mathrm{e}^{-T_d s} \tag{1}$$

is considered, where: k_1 - the plant gain, T_d - the time delay, s - the complex variable in the L-transform.

Further the D-transform is used. It is defined by the relations

$$X(\gamma) = \mathrm{D}\{x(kT)\} = T\sum_{k=0}^{\infty} x(kT)(T\gamma+1)^{-k} \tag{2}$$

$$x(kT) = \mathrm{D}^{-1}\{X(\gamma)\} = \frac{1}{2\pi j}\oint_C X(\gamma)(T\gamma+1)^{k-1}\mathrm{d}\gamma$$

where: $X(\gamma)$ - the D-transform of the original $x(kT)$, D and D^{-1} - the direct and inverse D-transform operators, γ - the complex variable in the D-transform, k - the relative discrete time $(k = 0, 1, 2, \ldots)$, T - the sampling period.

The closed integration path C lays within convergence domain of the transform $X(\gamma)$ and contains its all singular points. The conditions for the original are the same like for Z-transform. More detailed information can be found in publications [5, 6].

The simple transfer relations among complex variables, transforms and transfer functions hold:

complex variables

$$s = \lim_{T\to 0}\gamma, \quad z = T\gamma + 1 \tag{3}$$

transforms of variables

$$X(s) = \lim_{T\to 0} X(\gamma), \quad X(z) = \frac{1}{T}X(\gamma)\Big|_{\gamma=\frac{z-1}{T}} \tag{4}$$

transfer functions

$$G(s) = \lim_{T\to 0} G(\gamma), \quad G(z) = G(\gamma)|_{\gamma=\frac{z-1}{T}} \tag{5}$$

[1]*prof. Ing. Miluse Viteckova, CSc.*, VSB - Technical University of Ostrava, Czech Republic, Department of Control Systems and Instrumentation, miluse.viteckova@vsb.cz

[2]*prof. Ing. Antonin Vitecek, CSc., Dr.h.c.*, VSB - Technical University of Ostrava, Czech Republic, Department of Control Systems and Instrumentation, antonin.vitecek@vsb.cz

[3]*Ing. Marek Babiuch, Ph.D.*, VSB - Technical University of Ostrava, Czech Republic, Department of Control Systems and Instrumentation, marek.babiuch@vsb.cz

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where: $X(s)$ - the L-transform of original $x(t)$, $X(z)$ - the Z-transform of original $x(kT)$, $G(s)$ - the L-transfer function, $G(z)$ - the Z-transfer function, $G(\gamma)$ - the D-transfer function, t - continuous time, z - the complex variable in the Z-transform.

In the case of the D/A converter in a control system with a digital controller which corresponds to a zero-order hold is used the plant D-transfer function is given by the formula [5, 6].

$$G_P(\gamma) = \frac{\gamma}{T\gamma+1} \mathrm{D}\left\{ \mathrm{L}^{-1}\left\{ \frac{G_P(s)}{s} \right\}\Big|_{t=kT} \right\} \tag{6}$$

The D-transfer function of the plant (1) in accordance with (6) is given by relation

$$G_P(\gamma) = \frac{k_1}{\gamma}(T\gamma+1)^{-d} \tag{7}$$

where the relative discrete time delay

$$d = \frac{T_d}{T}$$

meanwhile the integer number is supposed.

The control system with the conventional PI controller (with one-degree-of-freedom) is shown in Fig. 1, where: W' - the desired variable transform, Y - the controlled variable transform, V - the disturbance variable transform, E - the error transform, G_C - the controller transfer function, G_P - the plant transfer function.

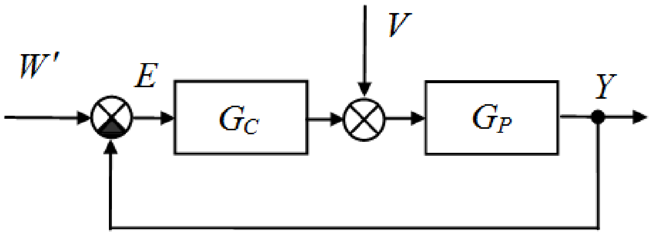

Fig. 1: Control system with conventional controller.

The error D-transfer function for the conventional PI controller

$$G_C(\gamma) = K_P\left(1 + \frac{T\gamma+1}{T_I\gamma}\right) \tag{8}$$

and the IPTD plant (7) is given by the formula

$$G_{w'e}(\gamma) = \frac{E(\gamma)}{W'(\gamma)} = \frac{T_I\gamma^2}{T_I\gamma^2 + k_1K_P[(T_I+T)\gamma+1](T\gamma+1)^{-d}} \tag{9}$$

where: K_P - the controller gain, T_I - the integral time.

For the unit step

$$W'(\gamma) = \frac{T\gamma+1}{\gamma}$$

the linear control area is given by the relation [5, 6].

$$T\sum_{k=0}^{\infty} e(kT) = \lim_{\gamma\to 0} E(\gamma) = \lim_{\gamma\to 0}[G_{w'e}(\gamma)W'(\gamma)] = 0 \tag{10}$$

The relation (10) has very important interpretation. It expresses the fact, that the areas under and over the half line determined by the desired value are equal, i.e. one overshoot appears in the servo response at least. Therefore it is impossible to tune the conventional PI controller so that the servo response is without the overshoot.

If the servo response without the overshoot is desired, then the use of the PI controller with two-degree-of-freedom (2DOF) is elegant solution [3, 4].

Two-Degree-of-Freedom Controller Tuning

The 2DOF PI controller in the γ-complex variable domain can be described by the relation

$$U(\gamma) = K_P\left\{bW(\gamma) - Y(\gamma) + \frac{T\gamma+1}{T_I\gamma}[W(\gamma)-Y(\gamma)]\right\} \tag{11}$$

which can be modify in the form (Fig. 2)

$$U(\gamma) = G_F(\gamma)G_C(\gamma)W(\gamma) - G_C(\gamma)Y(\gamma) \tag{12}$$

$$G_F(\gamma) = \frac{(bT_I+T)\gamma+1}{(T_I+T)\gamma+1} \tag{13}$$

where: $G_C(\gamma)$ - the conventional PI controller D-transfer function (8), $G_F(\gamma)$ - the input filter D-transfer function, b - the set-point weight for proportional term ($0 \le b \le 1$).

For $b = 1$ the relation (11) describes the conventional PI controller (8), $G_F = 1$ and $W = W'$.

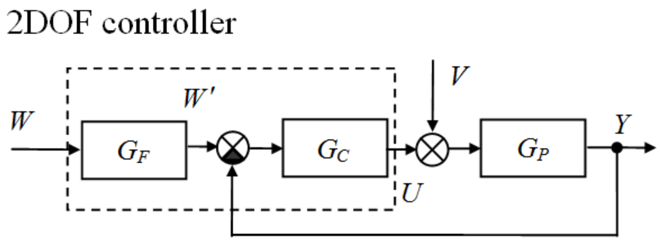

Fig. 2: Control system with 2DOF controller.

From Fig. 2 it is obvious that the conventional PI controller (8) must be tuned from the point of view of the desired regulatory response and the input filter (13) weight b must be chosen from the point of view of the desired servo response [3, 4].

In the article for both cases the multiple dominant pole method is used. This method supposes that the non-dominant poles and zeros have an insignificant influence on the control process. For more detail, see [7, 8].

For the conventional PI controller the characteristic polynomial of the control system in Fig. 1 has the form [see the denominator of the D-transfer function (9)]

$$N(\gamma) = T_I\gamma^2(T\gamma+1)^d + k_1K_P[(T_I+T)\gamma+1] \tag{14}$$

The triple dominant pole γ_3^* and the adjustable controller parameters T_I^* and K_P^* can be obtained by solution of the three equations

$$\frac{\mathrm{d}^iN(\gamma)}{\mathrm{d}\gamma^i} = 0 \quad \text{for } i = 0,1,2 \tag{15}$$

It is obtained

$$\gamma_3^*(T) = -\frac{1}{(d+2)T}\left(2 - \sqrt{\frac{2d}{d+1}}\right) \tag{16}$$

$$K_P^*(T) = -\frac{1}{k_1}[(d+1)T\gamma_3^*+2]\gamma_3^*(T\gamma_3^*+1)^d \tag{17}$$

$$T_I^*(T) = -\frac{(d+1)T^2(\gamma_3^*)^2+(d+3)T\gamma_3^*+2}{[(d+1)T\gamma_3^*+1]\gamma_3^*} \tag{18}$$

For the triple dominant pole (16) computation the positive + sign must be considered.

For the computed adjustable controller parameters (17) and (18) the control system D-transfer function can be approximated by the dominant pole (16)

$$G_{w'y}(\gamma) = \frac{Y(\gamma)}{W'(\gamma)} \approx \frac{(T_I^*+T)\gamma+1}{\left(\frac{1}{|\gamma_3^*|}\gamma+1\right)^3}(T\gamma+1)^{-d} \tag{19}$$

The big overshoot in the servo response is caused by the numerator of the D-transfer function (19) and therefore it is suitable to compensate for it.

The control system D-transfer function for the 2DOF PI controller in accordance with Fig. 2 and the relations (12), (13) and (19) has the form

$$G_{wy}(\gamma) = \frac{Y(s)}{W(s)} = G_F(\gamma)G_{w'y}(\gamma) \approx \frac{(bT_I^* + T)\gamma + 1}{(T_I^* + T)\gamma + 1} \frac{(T_I^* + T)\gamma + 1}{\left(\frac{1}{|\gamma_3^*|}\gamma + 1\right)^3}(T\gamma + 1)^{-d} \tag{20}$$

From the relation (20) follows that the input filter denominator compensates for the numerator of the control system D-transfer function. It causes the overshoot removing and simultaneously the servo response slowdown. By the suitable choice of the set-point weight b it is possible by the input filter numerator to compensate for one binomial of the denominator of the control system D-transfer function, i.e.

$$(bT_I^* + T)\gamma + 1 = \frac{1}{|\gamma_3^*|}\gamma + 1 \Rightarrow b^*(T) = \frac{1}{T_I^*}\left(\frac{1}{|\gamma_3^*|} - T\right) \tag{21}$$

The dynamics of the whole control system will be simpler and the servo response accelerates in this case.

The relations (16) - (18) and (21) directly hold for the conventional digital PI controller with the Z-transfer function

$$G_C(z) = K_P\left(1 + \frac{T}{T_I}\frac{z}{z-1}\right) \tag{22}$$

and the input filter with the Z-transfer function

$$G_F(z) = \frac{(bT_I + T)z - bT_I}{(T_I + T)z - T_I} \tag{23}$$

The corresponding relations for the conventional analog PI controller with the L-transfer function

$$G_C(s) = K_P\left(1 + \frac{1}{T_I s}\right) \tag{24}$$

and the input filter with the L-transfer function

$$G_F(z) = \frac{bT_I s + 1}{T_I s + 1} \tag{25}$$

can be obtained for the limit transition $T \to 0$.

It is obtained

$$s_3^* = \lim_{T\to 0}\gamma_3^*(T) = -\frac{2-\sqrt{2}}{T_d} \tag{26}$$

$$K_P^* = \lim_{T\to 0}K_P^*(T) = \frac{2(\sqrt{2}-1)}{k_1 T_d}\mathrm{e}^{\sqrt{2}-2} \doteq 0.461\frac{1}{k_1 T_d} \tag{27}$$

$$T_I^* = \lim_{T\to 0}T_I^*(T) = (3+2\sqrt{2})T_d \doteq 5.828T_d \tag{28}$$

$$b^* = \lim_{T\to 0}b^*(T) = \frac{2-\sqrt{2}}{2} \doteq 0.293 \tag{29}$$

The relations (16) - (18) and (21) for the digital 2DOF PI controller have unpleasant forms for practical use. They can be fundamentally simplified but it is somewhat inaccurate.

For the sufficient small sampling period the values for the digital controller can be approximated by the values for the corresponding analog controller if the time delay is increased to the half of the sampling period [2].

On the basis of the relations (27) - (29) it can be obtained

$$K_P^* \approx \frac{2(\sqrt{2}-1)}{k_1\left(T_d + \frac{T}{2}\right)}\mathrm{e}^{\sqrt{2}-2} \doteq \frac{0.461}{k_1\left(T_d + \frac{T}{2}\right)} \tag{30}$$

$$T_I^* \approx (3+2\sqrt{2})\left(T_d + \frac{T}{2}\right) \doteq 5.828\left(T_d + \frac{T}{2}\right) \tag{31}$$

For

$$T \le \frac{T_d}{2} \quad (d \ge 2) \tag{32}$$

the approximation of the controller gain (30) gives the error less than 3 % and approximation of the integral time (31) gives the error less than 5 %.

The set-point weight (29) can be approximated by the constant value (29) for

$$T \le \frac{T_d}{4} \quad (d \ge 4) \tag{33}$$

with the error less than 5 % or for (32) by the approximate relation

$$b^* \approx \frac{T_d}{(2+\sqrt{2})\left(T_d + \frac{T}{5}\right)} \doteq 0.293 \frac{T_d}{T_d + \frac{T}{5}} \tag{34}$$

which gives the error less than 1 %.

The simplified relations are universal, for $T > 0$ they hold for the digital 2DOF PI controller and for $T = 0$ hold for the analog 2DOF PI controller. It can see as well that the supposition that d must be an integer number is not substantial.

Example

It is necessary to tune the analog and digital 2DOF PI controller for the IPTD plant with the transfer function

$$G_P(s) = \frac{0.05}{s} \mathrm{e}^{-5s}$$

so that the servo and regulatory step responses were non-oscillatory without overshoots (the time delay is in seconds).

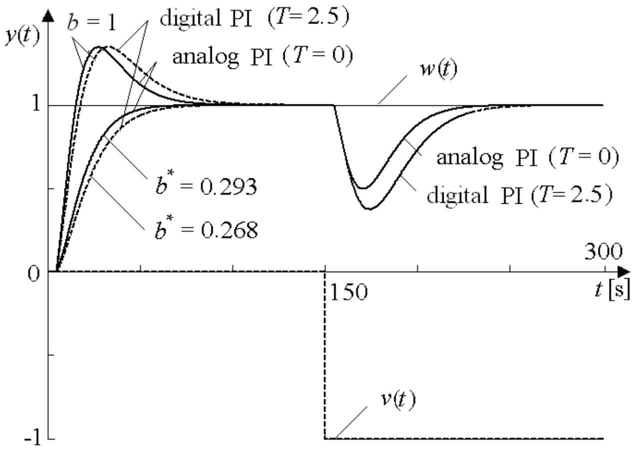

Fig. 3: Responses of control system with analog and digital conventional (b = 1) and 2DOF PI controller tuned by the multiple dominant pole method.

For the plant parameters $k_1 = 0.05\ \mathrm{s}^{-1}$ and $T_d = 5$ s, the sampling period $T = 0$ and $T = 2.5$ s there can be obtained:

Analog 2DOF PI controller

The relation (27) - (29) or (30), (31) and (34) for $T = 0$:

$$K_P^* \doteq 1.845, \quad T_I^* \doteq 29.142, \quad b^* \doteq 0.293$$

The responses of the control system with the analog 2DOF PI controller for $b^* \doteq 0.293$ and $b = 1$ (conventional) are shown in Fig. 3

Digital 2DOF PI controller

The accurate relations (16) - (18) and (21) for $T = 2.5 \Longrightarrow d = 2$:

$$\gamma_3^* \doteq -0.0845, \quad K_P^* \doteq 1.437, \quad T_I^* \doteq 34.821, \quad b^* \doteq 0.268$$

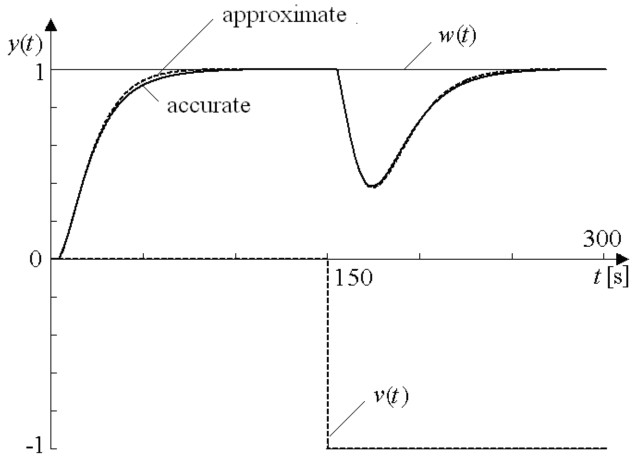

Fig. 4: Responses of control system with digital 2DOF PI controller tuned by the multiple dominant pole method.

The approximate relations (30), (31) and (34) for $T = 2.5$:

$$K_P^* \doteq 1.476, \quad T_I^* \doteq 36.428, \quad b^* \doteq 0.266$$

The responses of the control system with the digital 2DOF PI controller for the accurate values $b^* \doteq 0.268$ and $b = 1$ (conventional) are shown in Fig. 3 and for the accurate and approximate values in Fig. 4.

The high value of the sampling period $T = 2.5$ s $(d = 2)$ was specially chosen in order to show that the approximate relations (30), (31) and (34) are quite satisfactory for practical use.

Conclussion

The paper describes the unified approach to analog and digital 2DOF PI controller tuning by the multiple dominant pole method for the integrating plant with a time delay. The approach is fully derived. It ensures the servo and regulatory responses non-oscillatory without overshoot. The described tuning method is relatively robust and very simple and therefore there are good assumptions for its practical use.

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